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MULTIDIMENSIONAL CRYSTALLINE MEASURES

YVES MEYER

Abstract. One dimensional crystalline measures are well understood. In two or more dimensions the theory is still in its infancy. A “lighthouse” is a positive Radon measure on $\mathbb{R}^n$ which is supported by a closed set F of zero Lebesgue measure and whose Fourier transform is supported by a proper double cone. The pointwise product between n “independent lighthouses” is still a positive Radon measure. Under some natural conditions this product is a crystalline measure. This surprising result unifies the constructions of some two dimensional crystalline measures which were proposed in [9], [12], and [13].

1. Definition of crystalline measures

Let us fix some notations which are needed to define crystalline measures. The euclidean norm of $a \in \mathbb{R}^n$ is denoted by $|a|$. The Dirac measure at $a \in \mathbb{R}^n$ is denoted by δ_a or $\delta_a(x)$. It is defined by $\langle \delta_a, f \rangle = f(a)$ for any continuous function f. A measure is always a Radon measure in this note. If μ is a real valued and bounded Radon measure, we have $\mu = \mu_0 - \mu_1$ where μ_0 and μ_1 are mutually singular positive measures. The norm of μ is the sum between the total masses of μ_0 and μ_1 and is denoted by $\|\mu\|$. A *purely atomic measure* (or an atomic measure) is a series $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ of Dirac measures δ_λ at $\lambda \in \Lambda \subset \mathbb{R}^n$ where the coefficients $c(\lambda)$ are real or complex numbers and $\sum_{\{\lambda \in \Lambda; |\lambda| \le R\}} |c(\lambda)|$ is finite for any $R \ge 0$. A subset $\Lambda \subset \mathbb{R}^n$ is *locally finite* if $\Lambda \cap B$ is finite for any bounded set B. Equivalently Λ is *locally finite* if it can be ordered as a sequence $\{\lambda_j, j = 1, 2, \dots\}$ such that $|\lambda_j|$ tends to infinity with j. A locally finite set Λ is uniformly discrete if there exists a positive r such that for any $\lambda \in \Lambda$ and any $\lambda' \in \Lambda, \lambda' \neq \lambda$, we have $|\lambda' - \lambda| \ge r$. Such a set Λ is a Delone set if it is relatively dense. It means that there exists a positive R such that the balls centered at $\lambda \in \Lambda$ with radius R are a covering of $\mathbb{R}^n$. A measure μ is a *tempered distribution* if it has a polynomial growth

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at infinity in the sense given by Laurent Schwartz in [18]. For instance the measure $\sum_1^\infty 2^k\delta_k$ is not a tempered distribution while $\sum_1^\infty k^3\delta_k$ and $\sum_1^\infty 2^k[\delta_{(k+2^{-k})} - \delta_k]$ are tempered distributions. The *Fourier transform* $\mathcal{F}(f) = \widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix \cdot y) f(x)\, dx. \tag{1}$$

The *distributional Fourier transform* $\widehat{\mu}$ of a tempered distribution μ is defined by the following rule: for any test function φ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ one has $\langle \widehat{\mu}, \varphi \rangle = \langle \mu, \widehat{\varphi} \rangle$.

Definition 1. *An atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if the three following conditions are satisfied:*

- (a) *the measure μ is a tempered distribution,*
- (b) *the support of μ is a locally finite set.*
- (c) *the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set.*

The simplest example of a crystalline measure is the Dirac comb $\nu = \sum_{k\in\mathbb{Z}} \delta_k$. Then $\widehat{\nu} = \nu$ and for any function φ in the Schwartz class we have $\sum_{k\in\mathbb{Z}} \varphi(k) = \langle \nu, \varphi \rangle = \langle \nu, \widehat{\varphi} \rangle = \sum_{m\in\mathbb{Z}} \widehat{\varphi}(m)$ which is the standard Poisson summation formula. More generally for any lattice $\Gamma \subset \mathbb{R}^n$ the atomic measure $\nu_\Gamma = \sum_{\gamma\in\Gamma} \delta_\gamma$ is a crystalline measure and we have $\widehat{\nu}_\Gamma = c_\Gamma \nu_{\Gamma^*}$ where Γ^* is the dual lattice of Γ and $1/c_\Gamma$ is the volume of a fundamental domain of Γ. In this essay any translated of a Dirac comb is also called a Dirac comb. We have for any function φ in the Schwartz class $\langle \widehat{\mu}, \varphi \rangle = \langle \mu, \widehat{\varphi} \rangle$ and if μ is a crystalline measure this is a new Poisson summation formula. The collection of all crystalline measures on $\mathbb{R}^n$ is a vector space. The product $P(x)d\mu(x)$ between a finite trigonometric sum $P(x) = \sum_{y\in F} a(y)\exp(2\pi ix \cdot y)$ and a crystalline measure $d\mu(x)$ is still a crystalline measure.

Definition 2. *An atomic measure μ is a trivial crystalline measure if $\mu = \sum_1^N P_j\nu_j$ where each P_j, $1 \le j \le N$, is a finite trigonometric sum and each ν_j is a Dirac comb.*

The distributional Fourier transform of a trivial crystalline measure is a trivial crystalline measure. If μ is a crystalline measure and if $A \in SL(n,\mathbb{R})$ then the pushforward measure $\mu \circ A^{-1}$ is still a crystalline measure. Crystalline measures were introduced simultaneously by Andrew Guinand in [2] and by Jean-Pierre Kahane & Szolem Mandelbrojt in [4]. Kahane and Mandelbrojt were studying meromorphic functions $\phi(s)$ in the complex plane enjoying the following three properties: (a) there exists a real number s_0 and an increasing sequence of positive real numbers $0 < \lambda_1 < \lambda_2 < \cdots$ such that $\phi(s)$ is the sum of the Dirichlet series $\sum_1^\infty c_j\lambda_j^{-s}$ on $\Re s > s_0$, (b) either $\phi(s)$ is an entire function or $\phi(s)$ is a meromorphic function in the complex plane and the unique pole of $\phi(s)$ is $s = 1$, and (c) $\phi(s)$ satisfies the same functional equation as the Riemann zeta function. Kahane and Mandelbrojt proved the following theorem:

Theorem 1. *Let us assume that properties (a), (b), and (c) are satisfied by the meromorphic function $\phi(s) = \sum_1^\infty c_j\lambda_j^{-s}$ and that there exist a $\beta > 0$ and some intervals J_k whose lengths tend to infinity such that $\lambda_{j+1} - \lambda_j \ge \beta > 0$ for $j \in J_k$. Then there exists* *a constant* c *such that* $\mu = c\,\delta_0 + \sum_1^\infty c_j(\delta_{\lambda_j} + \delta_{-\lambda_j})$ *is a crystalline measure which satisfies* $\widehat{\mu} = \mu$.

From the results obtained by Nir Lev and Alexander Olevskii [6], [7], it can be conjectured that the only crystalline measures μ of Theorem 1 are generalized Dirac combs. Then the meromorphic function $\phi(s)$ of Theorem 1 reduces to the Riemann zeta function after some trivial modifications. The converse implication is much easier. Let us assume that $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ is a crystalline measure and that there exists a $s_0 > 0$ such that $\sum_{\lambda\in\Lambda,\lambda\neq 0} |c(\lambda)||\lambda|^{-s_0}$ is finite. A similar condition is imposed to $\widehat{\mu}$. Then the corresponding zeta function is defined as the sum of the Dirichlet series $\zeta(\mu, s) = \sum_{\{\lambda\in\Lambda,\lambda\neq 0\}} c(\lambda)|\lambda|^{-s}$, $s \in \mathbb{C}$, $\Re s > s_0$. This function $\zeta(\mu, s)$ is obviously analytic in the open half plane defined by $s \in \mathbb{C}$, $\Re s > s_0$. If $n = 1$ and if μ is the Dirac comb then $\zeta(\mu, s)$ is two times the Riemann zeta function. If μ is a n dimensional Dirac comb then $\zeta(\mu, s)$ is the Epstein zeta function. Kahane and Mandelbrojt made the following observation:

Theorem 2. *Let* $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ *be a crystalline measure on* $\mathbb{R}^n$ *and let* $\widehat{\mu} = \sum_{y\in F} a(y)\delta_y$ *be the distributional Fourier transform of* μ. *We consider the quantity* $\xi(\mu, s) = \pi^{-s/2}\Gamma(s/2)\zeta(\mu, s)$. *If the above mentioned properties are satisfied then* $\xi(\mu, s) - \frac{2a(0)}{n-s} - \frac{2c(0)}{s} = E(\mu, s)$ *is an entire function and we have* $\xi(\mu, s) = \xi(\widehat{\mu}, n-s)$ *identically on* $\mathbb{C} \setminus \{0, n\}$.

Kahane and Mandelbrojt observed that Theorem 2 can be traced back to Riemann and Titchmarsh [4]. If $n = 1$ and if $\widehat{\mu} = \mu$ then $\zeta(\mu, s)$ satisfies the same functional equation as the Riemann zeta function. The first non trivial crystalline measure satisfying $\widehat{\mu} = \mu$ was suggested by Guinand in [2]. Guinand defined a sequence γ_k, $k = 0, 1, \dots,$ of rational numbers by $\sum_0^\infty \gamma_k q^k = \prod_1^\infty (1-q^n)(1+q^{2n})^{2/3}(1+q^n)^{1/3}$ where $|q| < 1$. We have $\gamma_0 = 1, \gamma_1 = -2/3, \gamma_2 = -4/9, \gamma_3 = -40/81, \gamma_4 = -160/243, \gamma_5 = 268/729, \dots$ and $|\gamma_k| \leq Ck^{1/3}$. Let $\lambda_k = \sqrt{k + 1/9}$, $k = 0, 1, \dots$. Then Guinand's crystalline measure is $\mu_G = \sum_0^\infty \gamma_k(\delta_{\lambda_k} + \delta_{-\lambda_k})$. The Guinand measure μ_G satisfies $\widehat{\mu}_G = \mu_G$. This was announced in [2] and proved in [11]. Nir Lev and Alexander Olevskii gave other remarkable examples of non trivial crystalline measures [8]. Pavel Kurasov and Peter Sarnak constructed a non trivial crystalline measure which is a sum $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ of Dirac measures δ_λ on a Delone set $\Lambda \subset \mathbb{R}$ [5]. Maryna Viazovska could bridge the gap between crystalline measures and sharp sphere packings [17].

We now address n-dimensional crystalline measures. The tensor product $\mu = \mu_1 \otimes \dots \otimes \mu_n$ between n crystalline measures on the line is a crystalline measure on $\mathbb{R}^n$. Is it the unique way to construct crystalline measures on $\mathbb{R}^n$? A counter example is given in [9] for $n = 2$. But the crystalline measure μ which is constructed in [9] is periodic with respect to the first variable. It satisfies $\mu * \delta_{e_1} = \mu$ if $e_1 = (1, 0)$. The anonymous referee of [9] is wondering if one could get rid of this limitation. An answer is given in [13]. To our great surprise each of the crystalline measures of [9] and [13] is a pointwise product between two independent lighthouses. This observation is the motivation of this note. In full generality it is proved that a pointwise product between two independent lighthouses is a positive Radon measure. This is Theorem 5. Theorem 5 and its corollary provide us with a general scheme to construct crystalline measures. Theorem 5 is a simple mathematical fact since it follows from Hörmander's theorem on the pointwise product between two distributions. Therefore Hörmander's theorem is the Ariadne's thread of our construction and motivates the definition of a lighthouse. It remains to construct non trivial lighthouses. That is the place where Ahern measures are useful. They provide us with a simple recipe to construct lighthouses. Ahern measures are directly related to inner functions in the polydisc. Finally starting with a pair of two smooth inner functions in the unit disc one easily constructs two independent lighthouses on $\mathbb{R}^2$. Under a natural condition the pointwise product between these independent lighthouses is a crystalline measure.

2. Hörmander's theorem

The pointwise product $u\,v$ between two tempered distributions $u, v \in \mathcal{S}'(\mathbb{R}^n)$ does not exist in general. However this product exists if the two distributions u and v are independent in a sense given by two-microlocal analysis. This is Hörmander's theorem (Theorem 3). Here is a tentative definition of the pointwise product of two tempered distributions. Let φ and ψ be two functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\int \varphi = 1$, $\int \psi = 1$, and let φ_ϵ, $0 < \epsilon \leq 1$, be given by $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. The family ψ_ϵ, $0 < \epsilon \leq 1$, is defined similarly.

Definition 3. *Let u and v be two tempered distributions. Let us assume that (i) the pointwise product $(u * \varphi_\epsilon)(v * \psi_\epsilon)$ tends to a limit in $\mathcal{S}'(\mathbb{R}^n)$ as ϵ tends to 0 and (ii) that this limit does not depend on the choices of φ and ψ. Then the pointwise product uv between u and v exists and is defined by*

$$u\,v = \lim_{\epsilon \to 0} (u * \varphi_\epsilon)(v * \psi_\epsilon).$$

We now follow Hörmander's seminal work. The *singular support* $F_u \subset \mathbb{R}^n$ of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is a closed set which is defined by its complement $\mathbb{R}^n \setminus F_u$. This complement is the largest open set $\Omega \subset \mathbb{R}^n$ on which u coincides with a $\mathcal{C}^\infty$ function. The unit sphere of $\mathbb{R}^n$ centered at 0 is denoted by $\mathbb{S}^{n-1}$. The *wave front set* $WF(u)$ of u is a closed subset of $F_u \times \mathbb{S}^{n-1}$. Here again $WF(u)$ is defined by its complement in $F_u \times \mathbb{S}^{n-1}$. A pair (x_0, ξ_0), $x_0 \in F_u, |\xi_0| = 1$, does not belong to $WF(u)$ if and only if there exists (i) a compactly supported smooth function φ on $\mathbb{R}^n$ such that $\varphi(x_0) = 1$ and (ii) a circular cone $C_{\epsilon,\xi_0} \subset \mathbb{R}^n$ defined by $C_{\epsilon,\xi_0} = \{\xi; \xi \cdot \xi_0 \geq (1-\epsilon)|\xi|\}$ for some $\epsilon \in (0,1)$ and some $\xi_0 \in \mathbb{S}^{n-1}$ such that the Fourier transform of the product φu once restricted to C_{ϵ,ξ_0} is rapidly decreasing at infinity. Then $WF(u)$ is closed in $F_u \times \mathbb{S}^{n-1}$. If u is real valued, then $(x_0, \xi_0) \in WF(u)$ is equivalent to $(x_0, -\xi_0) \in WF(u)$. The wave front set $WF(u)$ of u is empty if and only if u is a $\mathcal{C}^\infty$ function. Many authors define the wave front set as a subset of $F_u \times (\mathbb{R}^n \setminus \{0\})$. These two viewpoints result in the same definition if a half line emanating from 0 is identified with its intersection with $\mathbb{S}^{n-1}$. A set $C \subset \mathbb{R}^n$ is a conic set if $x \in C$ and $\lambda \geq 0$ imply $\lambda x \in C$. Here is an obvious observation:

Remark 1. *Let $C \subset \mathbb{R}^n$ be a closed conic set and let u be a tempered distribution whose Fourier transform $\widehat{u}$ is supported by C. Then the wave front set $WF(u)$ of u is contained in $\mathbb{R}^n \times (C \cap \mathbb{S}^{n-1})$.*

Here is Hörmander's theorem ([3] p. 267):

Theorem 3. *Let u and v be two tempered distributions on $\mathbb{R}^n$. Let us assume that*

$$\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \quad (x,\xi) \in WF(u) \Rightarrow (x,-\xi) \notin WF(v).$$

Then the pointwise product $u\,v$ makes sense and is a tempered distribution.

If $F_u \cap F_v = \emptyset$ Hörmander's theorem is trivial: locally the product $u\,v$ reduces to the product between a distribution and a $\mathcal{C}^\infty$ function. Hörmander's sufficient condition does not allow us to defining $|u|^2$ if u is a distribution which is not a $\mathcal{C}^\infty$ function. Indeed $|u|^2 = u\overline{u}$ and if $(x,\xi) \in WF(u)$ we have $(x,-\xi) \in WF(\overline{u})$. Let us consider $u(x) = |x|$. Then $WF(u) = \{0\} \times \mathbb{S}^{n-1}$. However u^2 is $\mathcal{C}^\infty$. This example shows that Hörmander's sufficient condition is not necessary.

Corollary 1. *Let $C_1 \subset \mathbb{R}^n$ and $C_2 \subset \mathbb{R}^n$ be two closed conic sets such that $C_1 \cap (-C_2) = \{0\}$. Let u_1 and u_2 be two tempered distributions. If the Fourier transform of u_1 is supported by C_1 and the Fourier transform of u_2 is supported by C_2 then the pointwise product $u = u_1 u_2$ makes sense and the Fourier transform of u is supported by $C_1 + C_2$.*

This follows from Hörmander's theorem and Remark 1. Corollary 1 can be proved directly. Corollary 1 applies to the measures σ_A and σ_B which are constructed below. The product $\mu = \sigma_A \sigma_B$ is one of the crystalline measures we are looking for.

Another instance of Hörmander's theorem is given by the following example. We assume that:

- (i) $n = 2$, $\Gamma_1 \subset \mathbb{R}^2$ and $\Gamma_2 \subset \mathbb{R}^2$ are the graphs of two smooth functions $g_1 : \mathbb{R} \mapsto \mathbb{R}$ and $g_2 : \mathbb{R} \mapsto \mathbb{R}$,
- (ii) $\Gamma_1 \cap \Gamma_2$ is reduced to a single point $x_0 = (a,b)$,
- (ii) $g_2'(a) \neq g_1'(a)$ which implies that Γ_1 and Γ_2 are transverse at x_0,
- (iv) u is a measure supported by Γ_1, it is given by $u = \omega_1\, ds_{\Gamma_1}$ where ds_{Γ_1} is the arc length measure on Γ_1, and the density ω_1 of u is a smooth function,
- (v) v is defined similarly with respect to Γ_2.

The transversality between Γ_1 and Γ_2 at x_0 implies Hörmander's sufficient condition. Indeed the wave front set of u is contained in the set of pairs (x,ξ) such that $x \in \Gamma_1$ and ξ is normal to Γ_1 at x and the same property is valid for v. Therefore the product $u\,v$ makes sense. However Hörmander's theorem is not needed to define $u\,v$. A straightforward calculation yields the following result:

Lemma 1. *If u and v satisfy the above five properties then the product $u\,v$ makes sense and is given by*

$$u\,v = c\,\delta_{x_0}, \quad c = \omega_1(s_1(a))\omega_2(s_2(a))\frac{\sqrt{1+g_1'(a)^2}\sqrt{1+g_2'(a)^2}}{|g_2'(a) - g_1'(a)|} \tag{2}$$

where the values of the arc length on Γ_1 and Γ_2 at $x_0 = (a,b)$ are denoted by $s_1(a)$ and $s_2(a)$.

If Γ_1 and Γ_2 are orthogonal at x_0 and if $\omega_1 = \omega_2 = 1$ we have $g_1'(a)g_2'(a) = -1$ and (2) simply reduces to $u\,v = \delta_{x_0}$. Hörmander's theorem gives the right perspective on Lemma 1 since the transversality assumption $g_2'(a) \neq g_1'(a)$ of Lemma 1 is exactly

Hörmander's sufficient condition. Here is a sketch of the proof of Lemma 1. For easing the notations one assumes $\omega_1 = \omega_2 = 1$. For $\epsilon \in (0,1]$ one considers the function $u_\epsilon(x) = \epsilon^{-1}\sqrt{1+|g_1'(x_1)|^2}\chi_{u,\epsilon}(x)$ where $\chi_{u,\epsilon}$ is the indicator function of $\{x = (x_1,x_2);\ g_1(x_1) \le x_2 \le g_1(x_1)+\epsilon\}$. The distributional limit of $u_\epsilon(x)$ as ϵ tends to 0 is the arc measure ds_{Γ_1} on Γ_1. One defines $\chi_{v,\epsilon}(x)$ similarly. It now suffices to prove that $\lim_{\epsilon\to 0}\epsilon^{-2}\chi_{u,\epsilon}(x)\chi_{v,\epsilon}(x) = |g_2'(a)-g_1'(a)|^{-1}\delta_{x_0}$. Since g_1 and g_2 are smooth, it amounts to computing the area of the parallelogram which is defined by $\{x;\ g_1(a)+(x_1-a)g_1'(a) \le x_2 \le g_1(a)+(x_1-a)g_1'(a)+\epsilon\}$ and $\{x;\ g_2(a)+(x_1-a)g_2'(a) \le x_2 \le g_2(a)+(x_1-a)g_2'(a)+\epsilon\}$. This area is given by $|g_2'(a)-g_1'(a)|^{-1}\epsilon^2$ which ends the proof. Lemma 1 is used in our constructions of crystalline measures.

3. Lighthouses

Lighthouses are the building blocks of our construction of crystalline measures. Theorems 4 and 5 are the main tools of this construction. Under some natural assumptions the pointwise product between two independent lighthouses is a crystalline measure (Corollary 3 of Theorem 5). It remains to construct lighthouses. This is achieved in Section 4.

Lighthouses do not exist in one dimension. Lighthouses become interesting if the dimension n is larger than 1. A precise definition of a proper double cone is needed to define a lighthouse. In what follows a proper double cone C is a non empty set with the following properties:

(a) $C \subset \mathbb{R}^n$ is a closed set,
(b) $x \in C,\ t \in \mathbb{R}$, implies $tx \in C$,
(c) $C \neq \mathbb{R}^n$.

A double circular cone is defined by $C_{\epsilon,\xi_0} = \{x; |x\cdot\xi_0| \ge (1-\epsilon)|x|\}$ for some $\epsilon \in [0,1)$ and some unit vector $\xi_0 \in \mathbb{R}^n$. We begin with the definition of a lighthouse on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $|E|$ denote the Lebesgue measure of a Borel set $E \subset \mathbb{T}^n$. Here is our first definition of a lighthouse.

Definition 4. *A lighthouse on $\mathbb{T}^n$ is a pair (μ, S) consisting of a non negative Radon measure μ on $\mathbb{T}^n$ and of a proper double cone $S \subset \mathbb{R}^n$ such that*

(a) μ *is supported by a compact set K s.t. $|K| = 0$.*
(b)

$$\forall k \in \mathbb{Z}^n,\ k \notin S \Rightarrow \widehat{\mu}(k) = 0. \tag{3}$$

If (μ, S) is a lighthouse, then (μ, S') is still a lighthouse for any proper double cone S' containing S. If (μ, S) is a lighthouse there exists a smallest proper double cone T such that (μ, T) is a lighthouse. Indeed one denotes by E the support of $\widehat{\mu}$ and considers the set $F \subset \mathbb{R}^n$ consisting of all tx, $t \in \mathbb{R}, x \in E$. Since $E \subset S$ we obviously have $F \subset S$. Finally T is the closure of F. We have $T \subset S$ and T is a proper double cone which ends the proof. From now on we say that a non negative Radon measure μ is a lighthouse (S is not yet mentioned) if there exists a proper double cone S such that the pair (μ, S) satisfies (b). If it is the case the knowledge of S is not needed since the proper double cone T is fully determined by μ and (μ, T) satisfies (b).

Definition 5. *Let $H \subset \mathbb{R}^n$ be the proper double cone defined by $x = (x_1, \dots, x_n) \in H$ if either $x_1 \geq 0, \dots, x_n \geq 0$ or $x_1 \leq 0, \dots, x_n \leq 0$. If μ is a non negative Radon measure on $\mathbb{T}^n$ and if its Fourier coefficients vanish outside H then μ is called an Ahern measure.*

An Ahern measure on $\mathbb{T}^n$ which is supported by a compact set of measure 0 is a lighthouse. Ahern measures where introduced by P. R. Ahern in [1]. Ahern measures are implicit in the beautiful construction of crystalline measures achieved by Kurasov and Sarnak [5]. For any $A \in SL(n, \mathbb{Z})$ and any lighthouse (μ, S) the pair $((\mu \circ A^{-1}), (A^*)^{-1}(S))$ is still a lighthouse. Using this remark it is easy to arbitrarily move and shrink the proper double cone S around 0. This is why the name "lighthouse" is used. Here is a proof of this remark if $n = 2$ and if $S = H$. Let a and c two relatively prime integers. Bezout's theorem implies that two integers c and d exist such that $ad - bc = 1$. Let $A_N \in SL(2, \mathbb{Z})$ be defined by $(A_N^*)^{-1} = \begin{pmatrix} a & b + Na \\ c & d + Nc \end{pmatrix}$. Then the proper double cone $(A_N^*)^{-1}(H)$ is arbitrarily close to the line containing (a, c) if N is large enough. Once a lighthouse is constructed it yields a family of lighthouses with almost arbitrary proper double cones.

Definition 6. *Let $2 \leq m \leq n$. The proper double cones $S_1, \dots, S_m$ are independent if the following property holds: whenever m vectors $x(1) \in \mathbb{R}^n, \dots, x(m) \in \mathbb{R}^n$ satisfy*

$$x(j) \in S_j, \; x(j) \neq 0, \; 1 \leq j \leq m, \tag{4}$$

then these vectors $x(1), \dots, x(m)$ are linearly independent.

If $m = 2$ this is Hörmander's sufficient condition. If some unit vectors $\xi(j), 1 \leq j \leq m$, are linearly independent then the double circular cones $C_{\epsilon, \xi(j)}, 1 \leq j \leq m$, are independent when $\epsilon > 0$ is small enough. If $S_1, \dots, S_m$ are independent and if the proper double cones T_j satisfy $T_j \subset S_j$ then these T_j are still independent.

The following lemma is important for the construction of crystalline measures:

Lemma 2. *Let $2 \leq m \leq n$. If the proper double cones $S_1, \dots, S_m$ are independent then for any compact set $B \subset \mathbb{R}^n$ the mapping $P : (S_1 + B) \times \cdots \times (S_m + B) \mapsto \mathbb{R}^n$ which maps $(x_1, \dots, x_m)$ to $x_1 + \cdots + x_m$ is a proper map.*

We argue by contradiction. Let us assume that there exist a constant C and m sequences of vectors $x_{j,k} \in S_j, k \in \mathbb{N}, 1 \leq j \leq m$, such that $|x_{1,k} + \cdots + x_{m,k}| \leq C$ and $|x_{1,k}| + \cdots + |x_{m,k}| \to \infty$ with k. Let $q(k) \in [1, m]$ be defined by $|x_{q(k),k}| = \sup_j |x_{j,k}|$. We consider $x'_{j,k} = x_{j,k}/|x_{q(k),k}|$. Then $x'_{j,k} \in S_j$ and there exists a subsequence of the sequence $x'_{j,k}$ tending to z_j as k tends to ∞. We have $z_j \in S_j$ since S_j is closed. Finally $z_1 + \cdots + z_m = 0$ and $|z_q| = 1$ which contradicts the independence of the proper double cones. It implies the following result which is seminal in our construction of crystalline measures:

Corollary 2. *Let $2 \leq m \leq n$. If the proper double cones $S_1, \dots, S_m$ are independent, if $B \subset \mathbb{R}^n$ is a compact ball and if $E_j \subset S_j + B, 1 \leq j \leq m$, are m locally finite sets, then $E_1 + \cdots + E_m$ is a locally finite set.*

Definition 7. *Let $2 \leq m \leq n$. The lighthouses $(u_1, S_1), \dots, (u_n, S_m)$ are independent if the proper double cones $S_1, \dots, S_m$ are independent.*

Once more it is unnecessary to specify the proper double cones S_j in this definition since they can be replaced by the smallest proper double cones T_j. Our construction of crystalline measures relies on the generalization to $\mathbb{R}^n$ of the following theorem. This generalization is detailed in Theorem 5.

Theorem 4. *Let $2 \leq m \leq n$. Let $u_j, \dots, u_m$ be m independent lighthouses on $\mathbb{T}^n$. Then the pointwise product $\mu = u_1 \cdots u_m$ exists and is a positive measure. Moreover the support of μ is contained in the intersection $\bigcap_1^m K_j$ where K_j is the closed support of μ_j.*

This is still valid if $u_j, \dots, u_m$ are m weak lighthouses on $\mathbb{T}^n$ and are independent. The second assertion of Theorem 4 is obvious and does not use the spectral properties of lighthouses. Theorem 4 and its companion Theorem 5 are seminal to our construction of crystalline measures. Theorem 4 is not used in this note and Theorem 5 is only used for $m = 2$. If $m = 2$ Theorem 4 is an immediate corollary of Hörmander's theorem on the product between two distributions. Indeed if S and T are two independent proper double cones, if $\widehat{u}$ is supported by S, and if $\widehat{v}$ is supported by T then Hörmander's theorem implies that the pointwise product uv exists and is a distribution. If one is ready to accept that $u \geq 0$ and $v \geq 0$ imply $uv \geq 0$ the proof ends by the fact that a positive distribution is a measure. We denote the Fejer kernel in two variables by $G_N(x)$. To prove that $u \geq 0$ and $v \geq 0$ imply $uv \geq 0$ we consider $u_N = G_N * u$ and $v_N = G_N * v$. Then u_N and v_N are two non negative trigonometric polynomials. Therefore the pointwise product $u_N v_N$ is a non negative trigonometric polynomial. We have $u_N \rightharpoonup u$ in the distributional sense and $\widehat{u}_N$ is supported by S. Similarly $v_N \rightharpoonup v$ and $\widehat{v}_N$ is supported by T. Then the proof of Hörmander's theorem yields $u_N v_N \rightharpoonup uv$ in the distributional sense. Therefore uv is positive. It ends the proof. If the definition of a lighthouse (μ, S) was modified and if the measure μ whose Fourier transform is supported by a double cone S was allowed to be a signed measure then the pointwise product $\mu = u_1 \cdots u_m$ between m independent lighthouses would not be a signed measure in general. This product would exist by Hörmander's theorem but would be a tempered distribution (see Section 8).

A pedestrian proof of Theorem 4 which does not use Hörmander's theorem is given in Section 9. If $(u_j, S_j), 1 \leq j \leq m$, are m independent lighthouses then the total mass of μ is the product of the total masses of the u_j. If each $u_j, 1 \leq j \leq n$, is a probability measure, so is μ. If the proper double cones S_j are not independent the pointwise product $u_1 \cdots u_n$ does not exist even in the distributional sense. Theorem 4 also fails if the number of independent lighthouses is larger than n. As it was already mentioned Theorem 4 fails miserably if u_j are signed measures as it will be shown in Section 8. Let us give the simplest example of Theorem 4. Let $\Gamma_j \subset \mathbb{T}^n$ be defined by $\Gamma_j = \{x = (x_1, \dots, x_n) \in \mathbb{T}^n; x_j = 0\}$ and let u_j be the Haar measure on Γ_j. Let $S_j \subset \mathbb{R}^n$ be defined by $S_j = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_i = 0 \ \ \forall i \neq j\}$. Then $\widehat{u}_j(k) = 1$ if $k \in S_j \cap \mathbb{Z}^n$ while $\widehat{u}_j(k) = 0$ elsewhere. Therefore (u_j, S_j) is a trivial lighthouse and these $(u_j, S_j), 1 \leq j \leq n$, are independent lighthouses. Then the pointwise product $u_1 \cdots u_n$ is the Dirac measure δ_0.

We now define lighthouses on $\mathbb{R}^n$.

Definition 8. *A lighthouse on $\mathbb{R}^n$ is a pair (μ, S) consisting of a positive measure μ on $\mathbb{R}^n$ and of a proper double cone $S \subset \mathbb{R}^n$ such that:*

- (i) *The measure μ is a tempered distribution.*
- (ii) *The measure μ is supported by a closed set F such that $|F| = 0$.*
- (iii) *The distributional Fourier transform $\widehat{\mu}$ of μ is supported by S.*

Let us observe that (i) is equivalent to the following growth condition: there exists a constant C and an exponent q such that $\mu(B_R) \leq CR^q$ for any $R \geq 1$ where B_R is the ball of radius R centered at 0. Here again if (μ, S) is a lighthouse and if a double cone S' contains S then (μ, S') is also a lighthouse. If (μ, S) is a lighthouse there exists a smallest double cone T such that (μ, T) is a lighthouse. As above a lighthouse μ will often be defined as a non negative Radon measure on $\mathbb{R}^n$ supported by a closed set F of Lebesgue measure zero and such that there exists a double cone $S \subset \mathbb{R}^n$ for which (μ, S) is a lighthouse. As it was said above the smallest such double cone is uniquely determined by μ. A weak lighthouse is defined similarly: the positive measure μ is supported by a closed set F s.t. $|F| = 0$ and there exists a compact set K such that the distributional Fourier transform $\widehat{\mu}$ of μ is supported by $S \cup K$. The simplest example of a lighthouse on $\mathbb{R}^n$ is given by a lighthouse on $\mathbb{T}^n$ viewed as a $\mathbb{Z}^n$ periodic measure. The distributional Fourier transform of a lighthouse μ can be an atomic measure, it is the case allover in this essay. It can also be a tempered distribution but surprisingly it cannot be a continuous function on $\mathbb{R}^n$.

The total mass of a lighthouse (μ, S) on $\mathbb{R}^n$ is infinite. Indeed if μ was a bounded measure we would have $\widehat{\mu}(0) = c > 0$. Since $\widehat{\mu}$ is a continuous function it implies $|\widehat{\mu}(y)| \geq c/2$ if $|y| \leq r$ for some $r > 0$. This prevents $\widehat{\mu}$ from being supported by the double cone S. The definition of independent lighthouses is the same as in the periodic case.

Here is a simple example of a two dimensional lighthouse.

Lemma 3. *Let $E \subset \mathbb{R}$ be a locally finite set and $\sigma = \sum_{u \in E} c(u)\delta_u$ be a positive atomic measure supported by E. Let λ be the Lebesgue measure on the line. Then $\mu = \sigma \otimes \lambda$ is a two dimensional lighthouse.*

The measure μ is supported by a countable union F of vertical lines. We have $F = \{x = (x_1, x_2); x_1 \in E\}$ and $|F| = 0$. The support of the distributional Fourier transform of μ is contained in the horizontal axis. The horizontal axis of $\mathbb{R}^2$ is a trivial double cone. It ends the proof. We now reach the main result of this paper:

Theorem 5. *Let $2 \leq m \leq n$. Let $(u_j, S_j), \dots, (u_m, S_m)$ be m independent lighthouses on $\mathbb{R}^n$. Then the pointwise product $\mu = u_1 \cdots u_m$ is a positive measure. The support of μ is contained in the intersection $\cap_1^m F_j$ where F_j is the closed support of μ_j. Moreover if each $\widehat{u}_j$, $1 \leq j \leq m$, is an atomic measure supported by a locally finite set then the same is true for $\widehat{\mu}$.*

This is still valid if $(u_j, S_j), \dots, (u_m, S_m)$ are m independent weak lighthouses on $\mathbb{R}^n$. Here again if the definition of a lighthouse (μ, S) was modified and if the measure μ whose Fourier transform is supported by a double cone S was allowed to be a signed measure then the pointwise product $\mu = u_1 \cdots u_m$ between m independent lighthouse measures would not be a signed measure in general. This product would exist by Hörmander's theorem as a tempered distribution (see Section 8).

Corollary 3. *Let $(u_1, S_1), \dots, (u_m, S_m)$ be m independent lighthouses on $\mathbb{R}^n$ such that each $\widehat{u}_j, 1 \leq j \leq m$, is an atomic measure supported by a locally finite set. If $\mu = u_1 \cdots u_m$ is an atomic measure supported by a locally finite set then μ is a crystalline measure.*

As it was announced in the abstract all our previous constructions of two dimensional crystalline measures follow from this corollary. Here is a trivial example of Theorem 5. Let σ and τ be two positive crystalline measures on the line. With the notations of Lemma 3 let $\mu = \sigma \otimes \lambda$ and $\nu = \lambda \otimes \tau$. Then μ and ν are two independent lighthouses and we have $\mu\nu = \sigma \otimes \tau$. Trivial examples of two dimensional crystalline measures follow from Theorem 5. Up to some technicalities the proof of Theorem 5 is the same as the one of Theorem 4. We limit the discussion to $m = 2$. Let us assume that (u, S) and (v, T) are two independent lighthouses. Then Hörmander's theorem implies that the pointwise product uv is a tempered distribution. We now use the same argument as in Theorem 4 and conclude that this distribution is a positive measure which ends the proof. A pedestrian proof is given in Section 9. The second assertion of Theorem 5 follows from Lemma 2's corollary.

4. Inner functions and Ahern measures

The goal of this section is to relate Ahern measures to inner functions and to use this connection in the construction of Ahern measures. Once an Ahern measure is constructed it will be moved and rotated to obtain independent lighthouses. These lighthouses are the building blocks of our construction of crystalline measures (Sections 5 and 6). A leisurely promenade leading to Theorem 8 is proposed in this section. This pleasant promenade can be replaced by a short cut and the reader can move directly to Theorem 8. Indeed our construction of Ahern measures relies on Theorem 8 and does not depend on Theorem 6 and Theorem 7. Finally Theorem 8 is given a trivial proof at the end of this section.

The open polydisc $\mathbb{D}^m \subset \mathbb{C}^m$ is defined by $|z_j| < 1$, $1 \leq j \leq m$. The m dimensional torus $\mathbf{T}^m$ is defined by $|z_j| = 1$, $1 \leq j \leq m$. Then $\mathbf{T}^m$ is the distinguished boundary of $\mathbb{D}^m$. We have $\mathbb{T}^m \simeq \mathbf{T}^m$ and this isomorphism is given by the canonical map defined by $\theta = (\theta_1, \dots, \theta_m) \mapsto z = (\exp(2\pi i\theta_1), \dots, \exp(2\pi i\theta_m))$. P.R. Ahern and Walter Rudin proved that Ahern measures are related to inner functions in the polydisc [1], [15]. Let μ be a positive singular Radon measure on $\mathbf{T}^m$. Then Ahern wrote: *If the Poisson integral of μ is the real part of an holomorphic distribution F in the polydisc then $G(z) = \exp(-F(z))$ is a singular inner function in the polydisc and any singular inner function in the polydisc is obtained from this construction.* We give up this road and construct Ahern measures using a completely different approach which is described now. The algebra of bounded holomorphic function in $\mathbb{D}^m$ is denoted by $H^\infty(\mathbb{D}^m)$. If $f \in H^\infty(\mathbb{D}^m)$, then its trace $\lim_{r\uparrow 1} f(rz) = f(z)$ exists for almost all $z \in \mathbf{T}^m$. A function $f \in H^\infty(\mathbb{D}^m)$ is often identified to its trace on $\mathbf{T}^m$ and we then write $f \in H^\infty(\mathbf{T}^m)$.

Definition 9. *We say that $f \in H^\infty(\mathbb{D}^m)$ is an inner function if its trace on $\mathbf{T}^m$ satisfies $|f(z)| = 1$ almost everywhere. A constant of modulus 1 is a trivial inner function.*

The space of distributions on $\mathbf{T}^m$ is denoted by $\mathcal{D}'(\mathbf{T}^m)$. The Fourier coefficients $\widehat{\tau}(k)$, $k \in \mathbb{Z}^m$, of $\tau \in \mathcal{D}'(\mathbf{T}^m)$ have a polynomial growth at infinity.

Definition 10. *Let $S_0^+ \subset \mathbb{Z}^m$ be defined by $k_1 \geq 0, \dots, k_m \geq 0$. A distribution $\tau \in \mathcal{D}'(\mathbf{T}^m)$ is holomorphic if $\widehat{\tau} = 0$ on $\mathbb{Z}^m \setminus S_0^+$.*

If F is holomorphic in $\mathbb{D}^m$ and $0 \leq r < 1$ we define F_r on $\mathbf{T}^m$ by $F_r(z) = F(rz)$. Then $\tau \in \mathcal{D}'(\mathbf{T}^m)$ is holomorphic if and only if there exists a holomorphic function F in $\mathbb{D}^m$ such that $\lim_{r\uparrow 1} F_r = \tau$ in the distributional sense. This yields a new definition of Ahern measures.

Definition 11. *A positive measure on $\mathbf{T}^m$ is a Ahern measure if it is the real part of a holomorphic distribution.*

As above $S_0^- \subset \mathbb{Z}^m$ is defined by $k_1 \leq 0, \dots, k_m \leq 0$ and $H = S_0^+ \cup S_0^-$.

Lemma 4. *Let $\tau \in \mathcal{D}'(\mathbf{T}^m)$ be a holomorphic distribution. Let τ_0 be the real part of τ and τ_1 the imaginary part. Then $\widehat{\tau}_0 = 0$ on $\mathbb{Z}^m \setminus H$, the same is true for $\widehat{\tau}_1$, $\widehat{\tau}_1 = -i\widehat{\tau}_0$ on S_0^+, and $\widehat{\tau}_1 = i\widehat{\tau}_0$ on on S_0^-.*

This well known lemma follows from Definition 11 and from the hermitian symmetry of $\widehat{\tau}_0$.

Definition 12. *Let ν be a positive Radon measure on $\mathbf{T}$ and let $\sum_{k\in\mathbb{Z}} c_k z^k$ be the Fourier series expansion of ν. Let J be a non trivial inner function on $\mathbf{T}^m$. Then the Ahern measure $\nu \circ J$ is defined by*

$$\nu \circ J = \sum_{k\in\mathbb{Z}} c_k J^k. \tag{5}$$

It is shown now that if $r \in [0,1), r \to 1$, the function $\sum_{k\in\mathbb{Z}} c_k r^{|k|} J^k$ converges in the distributional sense to a Ahern measure on $\mathbf{T}^m$. This is given by the proof of the following theorem:

Theorem 6. *Let ν be a positive Radon measure on $\mathbf{T}$ and let J be a non trivial inner function on $\mathbf{T}^m$. Then $\nu \circ J$ is an Ahern measure on $\mathbf{T}^m$ and*

$$\|\nu \circ J\| \leq \frac{1 + |J(0)|}{1 - |J(0)|}\|\nu\|. \tag{6}$$

More generally let ν be an Ahern measure on the polydisc $\mathbf{T}^q$ and let $J_1, \dots, J_q$ be q non trivial inner functions on $\mathbf{T}^m$. We set $J = (J_1, \dots, J_q) : \mathbf{T}^m \mapsto \mathbf{T}^q$. Then $\nu \circ J$ is an Ahern measure on $\mathbf{T}^m$ and

$$\|\nu \circ J\| \leq (\frac{1 + |J_1(0)|}{1 - |J_1(0)|}) \cdots (\frac{1 + |J_q(0)|}{1 - |J_q(0)|})\|\nu\|. \tag{7}$$

Theorem 6 yields a definition and an estimate. Let us begin with the estimate. If $J(0) = 0$ it is given by the following lemma which is a classical result by Charles Loewner.

Lemma 5. *Let J be an inner function on $\mathbf{T}^m$ such that $J(0) = 0$ and let g be a continuous function on $\mathbf{T}$. Let $d\lambda_m$ be the Haar measure on $\mathbf{T}^m$. Then*

$$\int_{\mathbf{T}^m} g \circ J \, d\lambda_m = \int_{\mathbf{T}} g \, d\lambda_1 . \tag{8}$$

By linearity and density it suffices to prove (8) if $g(z) = z^k$, $k \in \mathbb{Z}$. If $k \geq 1$ we have $\int_{\mathbf{T^m}} J^k d\lambda_m = J(0)^k = 0$. If $k \leq -1$ we have $J^k = \overline{J}^{|k|}$ and $\int_{\mathbf{T^m}} J^k d\lambda_m = 0$. Finally (8) is trivial if $k = 0$. Lemma 5 is proved. It yields (6) if $J(0) = 0$ and if ν is absolutely continuous with respect to $d\lambda_1$ with a continuous density. If $m = 1$ and if $J(0) \neq 0$ we follow Aline Bonami and write $J = h \circ J_0$ where $h(z) = \frac{J(0)-z}{1-z\overline{J}(0)}$ and $J_0(0) = 0$. Then (6) follows from (8) applied to $g \circ h$. Finally Theorem 6 is proved if $m = 1$.

We return to Theorem 6 and prove the first assertion. We denote by $\nu = \sum_{k\in\mathbb{Z}} c_k z^k$ the Fourier series expansion of ν. For $0 \leq r < 1$ we denote by $P_r(z) = \frac{1-r^2}{|1-rz|^2}$ the Poisson kernel. Let $\nu(r, z) = \nu * P_r$. We have $\nu(r, z) = \sum_{k\in\mathbb{Z}} r^{|k|} c_k z^k$, $z \in \mathbf{T}$. Since ν is a positive measure and P_r is a positive kernel we have $\nu(r, z) \geq 0$. We set $\nu_J(r, z) = \sum_{k\in\mathbb{Z}} r^{|k|} c_k J^k(z)$. Since $0 \leq r < 1$ this series converges uniformly on $\mathbf{T}^m$. Indeed $|c_k| \leq \|\nu\|$ and $\|J\|_\infty = 1$. For almost every $z \in \mathbf{T}^m$ we have $|J(z)| = 1$ since J is an inner function. Therefore $\nu_J(r, z) = \nu(r, J(z))$ which implies $\nu_J(r, z) \geq 0$. Let $d\lambda_m$ be the Haar measure on $\mathbf{T}^m$. We have

$$\int_{\mathbf{T}^m} \nu_J(z, r) d\lambda_m = \sum_{k\in\mathbb{Z}} r^{|k|} c_k I_k \tag{9}$$

where $I_k = \int_{\mathbf{T^m}} J^k d\lambda_m = J(0)^k$ if $k \geq 0$ and $I_k = \overline{J}(0)^{|k|}$ if $k \leq -1$. But we have $|J(0)| < 1$ otherwise J would be a constant. This yields

$$\int_{\mathbf{T}^m} \nu_J(z, r) d\lambda_m \leq \frac{1 + |J(0)|}{1 - |J(0)|} \|\nu\|. \tag{10}$$

This uniform estimate implies that there exists a sequence $r_j \to 1$ such that $\nu_J(z, r_j)$ converges weakly to a positive measure on $\mathbf{T}^m$. We denote by $\nu \circ J$ this positive measure and prove that it does not depend on the choice of the sequence r_j. Let us consider the holomorphic function $F(z, r) = \sum_{k\in\mathbb{N}} c_k r^k J(z)^k$. We have $F(z, r) \in H^\infty(\mathbf{T^m})$ and $\nu_J(z, r) = 1 + 2\Re F(z, r)$. Then it suffices to prove that $F(z, r)$ converges in the distributional sense as $r \to 1$. Since the L^1 norms of $\nu_J(z, r)$, $r \in [0, 1)$, do not exceed $\frac{1+|J(0)|}{1-|J(0)|}\|\nu\|$ Lemma 4 implies that the set consisting of the functions $F(z, r)$, $r \in [0, 1)$, is bounded in the space $\mathcal{D}'(\mathbf{T}^m)$. Then it suffices to study the convergence of $F(z, r)$ in the polydisc to conclude to the convergence in the space $\mathcal{D}'(\mathbf{T}^m)$. Now $z \in \mathbf{T}$ is replaced by $\zeta \in \mathbb{D}^m$. We have $|J(\zeta)| < 1$ otherwise J would be a constant. Then $F(\zeta, r) = \sum_{k\in\mathbb{N}} c_k r^k J(\zeta)^k$ converges to $\sum_{k\in\mathbb{N}} c_k J(\zeta)^k$ as $r \to 1$. It ends the proof. The proof of the second assertion is similar.

Corollary 4. *If $f \in H^\infty(\mathbf{T}^m)$ is a non trivial inner function then the limit $\lim_{r\uparrow 1} \frac{1+rf}{1-rf}$ exists in the distributional sense. This limit is a holomorphic distribution $\tau \in \mathcal{D}'(\mathbf{T}^m)$.*

The real part of τ *is an Ahern measure on* $\mathbf{T}^m$ *which is denoted by* μ_f*. We then write*

$$\mu_f = \Re\,(\frac{1+f}{1-f}). \tag{11}$$

Moreover we have

$$\|\mu_f\| = \Re\,(\frac{1+f(0)}{1-f(0)}). \tag{12}$$

This is Theorem 6 with $\nu = \delta_0$.

The main ingredient in the construction of crystalline measures of Sections 6, 7 and 8, is an Ahern measure supported by a smooth hypersurface. It raises the following question: can $\nu \circ J$ be a singular measure? The measure $\nu \circ J$ is defined by Theorem 6.

Conjecture 1. *Let* $J \in H^\infty(\mathbf{T}^m)$ *be a non trivial inner function. Then the measure* ν_J *of Theorem 6 is singular with respect to the Haar measure on* $\mathbf{T}^m$ *if and only if* ν *is singular with respect to the Haar measure on* $\mathbf{T}$.

This is true if ν is supported by a compact set of measure 0 and if J is smooth as it is proved now. A function f belongs to the polydisc algebra $A(\mathbb{D}^m)$ if f is continuous on the closure of $\mathbb{D}^m$ and holomorphic on $\mathbb{D}^m$. As above a function in the polydisc algebra will be identified to its trace on $\mathbf{T}^m$. Walter Rudin and E.L. Stout proved that any inner function $f \in A(\mathbb{D}^m)$ is a rational function: $f = Q/P$ where P and Q are two polynomials and P does not vanish on $\mathbb{D}^m$. Moreover we have $Q(z) = M(z)P^*(1/z)$ where M is a monomial and the coefficients of the polynomial P^* are conjugates of the coefficients of P ([5]). Finally $1/z = (1/z_1, \dots, 1/z_m)$.

Lemma 6. *Let* J *be a smooth inner function and let* K *be the compact support of* ν*. Then* $\nu \circ J$ *is supported by* $U = \{x; J(x) \in K\}$.

This is given by the proof of Theorem 6. We now end the proof of the conjecture in the smooth case. If f is a continuous inner function then the set $U = \{x; f(x) \in K\}$ is a closed subset of $\mathbb{T}^m$. Moreover Theorem 6 implies that the Lebesgue measure $|U|$ of U is 0. Indeed let us define a function g_ϵ on $\mathbf{T}$ by $g_\epsilon(z) = 1$ on $K, 0 \le g_\epsilon \le 1$, and $g_\epsilon(z) = 0$ if the distance from z to K exceeds ϵ. Then the integral $\int g_\epsilon$ tends to 0 with ϵ. Theorem 6 implies that the Lebesgue measure of U does net exceed $C\epsilon$. This holds for any positive ϵ and $|U| = 0$. Therefore $\nu \circ H$ is singular with respect to the Haar measure on $\mathbb{T}^m$.

Let $f \in C^\infty(\mathbb{T}^m)$ be an inner function. Let $\phi(x)$, $x \in \mathbb{T}^m$, be the phase of $f(x)$. More precisely $\phi(x)$ is defined as a real valued continuous function on $\mathbb{R}^m$ such that

$$f(x) = \exp(2\pi i \phi(x)),\ x \in \mathbb{R}^m. \tag{13}$$

Then there exists a $q \in \mathbb{Z}^m$ such that ϕ satisfies the functional equation:

$$\phi(x+k) = \phi(x) + q \cdot x,\ \ k \in \mathbb{Z}^n. \tag{14}$$

In other terms $\phi(x) = q \cdot x + \psi(x)$ where ψ is a real valued $\mathbb{Z}^m$-periodic function. Finally ϕ is a C^∞ function. We now identify $\mathbf{T}^m$ with $\mathbb{T}^m$ and observe that ϕ can also be viewed as a $\mathbb{T}$ valued smooth function defined on $\mathbb{T}^m$. A simple corollary of Theorem 6 is the following result:

Theorem 7. *Let $f \in \mathcal{C}^\infty(\mathbb{T}^m)$ be an inner function. Let $x_0 \in \mathbb{T}^m$ and $b = f(x_0)$. Let us assume that $\nabla f(x) \neq 0$ everywhere on the level set $U_b = \{f(x) = b\}$. Then we have*

(a) *The limit $\lim_{r\uparrow 1} \frac{b+rf}{b-rf}$ exists in the distributional sense and is a holomorphic distribution denoted by $\frac{b+f}{b-f}$.*

(b) *$\Re(\frac{b+f}{b-f}) = \mu_f$ is an Ahern measure carried by the level set*

$$U_b = \{f(x) = b\}. \tag{15}$$

(c) *The measure μ_f is absolutely continuous with respect to the surface measure $d\sigma_{U_b}$ on U_b and we have $\mu_f = |\nabla f|^{-1}\, d\sigma_{U_b}$.*

To prove Theorem 7 it suffices to use Theorem 6 with $\nu = \delta_b$. Instead of proving (c) in full generality, let us treat an example which suffices to achieve our program.

Theorem 8. *Let f be a smooth inner function on $\mathbb{T}^m$. Let $\phi : \mathbb{R}^m \mapsto \mathbb{T}$ be the phase of f as defined by (13) and (14). Let $U_f \subset \mathbb{T}^{m+1}$ be the graph of the function $-\phi : \mathbb{T}^m \mapsto \mathbb{T}$. Let μ_f be the measure on $\mathbb{T}^{m+1}$ which is the image of the Haar measure on $\mathbb{T}^m$ by the mapping $x \mapsto (x, -\phi(x))$. Then μ_f is an Ahern measure on $\mathbb{T}^{m+1}$.*

To prove Theorem 8 we consider the auxiliary inner function $\widetilde{f}$ defined on $\mathbb{T}^{m+1}$ by

$$\widetilde{f}(x_1, \dots, x_m, x_{m+1}) = \exp(2\pi i x_{m+1}) f(x_1, \dots, x_m). \tag{16}$$

If complex coordinates were used $\widetilde{f}$ would be defined on $\mathbf{T}^{m+1}$ by

$$\widetilde{f}(z_1, \dots, z_m, z_{m+1}) = z_{m+1} f(z_1, \dots, z_m).$$

Properties (a) and (b) are already known if f is a continuous inner function. After multiplying f by $\overline{b}$ we can then assume $b = 1$. For proving Theorem 8 it suffices to use Theorem 6 with $\widetilde{f}$ instead of f and $\mathbb{T}^{m+1}$ instead of $\mathbb{T}^m$. Let $P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$, $0 \leq r < 1$, be the Poisson kernel. Then

$$\Re(\frac{1+r\widetilde{f}}{1-r\widetilde{f}}) = P_r(x_{m+1} + \phi(x_1, \dots, x_m)). \tag{17}$$

But the weak limit, as $r \to 1$, of the Poisson kernel P_r is the Dirac measure at 0. It implies that the weak limit as $r \to 1$ of the functions $P_r(x_{m+1} + \phi(x_1, \dots, x_m))$ is precisely the measure μ_f. Now Theorem 6 can be applied and μ_f is an Ahern measure on $\mathbb{T}^{m+1}$.

Here is a direct and simple proof of Theorem 8. Let us assume that $\exp(-2\pi i \phi)$ is an inner function and prove that μ_f is an Ahern measure. We have

$$\widehat{\mu}_f(k) = \int_{\mathbb{T}^{m-1}} \exp(-2\pi i k_m \phi) \exp(-2\pi i (k_1 x_1 + \cdots + k_{m-1} x_{m-1})) dx.$$

We know that the function $U = \exp(-2\pi i \phi)$ is an inner function. Therefore $\exp(-2\pi i k_m \phi) = U^{k_m}$ is also an inner function if $k_m \geq 0$. It implies $\widehat{\mu}_f(k) = 0$ unless $k_1 \geq 0, \dots, k_{m-1} \geq 0$. Since $\widehat{\mu}_f(-k) = \overline{\widehat{\mu}_f(k)}$ the proof is complete. The converse implication is as easy.

5. The first construction of a two dimensional crystalline measure

In this section it is shown that the construction of two dimensional crystalline measures of [13] follows from Theorem 5. As it was done in [13] we start with a building block $\sigma_r, (0 < r < 1)$ which is an Ahern measure on $\mathbb{T}^2$. Then this measure can also be viewed as a $\mathbb{Z}^2$-periodic measure on $\mathbb{R}^2$. Once σ_r is constructed we move it around 0 using two automorphisms A and B of $\mathbb{R}^2$ and obtain two independent lighthouses σ_A and σ_B on $\mathbb{R}^2$. Then we check that the pointwise product $\mu = \sigma_A \sigma_B$ is a sum of Dirac measures on a Delone set. Finally Theorem 5 implies that this product is a crystalline measure. Here are the details of this construction.

Let $r \in [0, 1)$. Let $\phi_r : \mathbb{R} \mapsto \mathbb{R}$ be the odd and decreasing function of the real variable θ defined by $\phi_r(0) = 0$ and $\frac{d\phi_r}{d\theta} = -\frac{1-r^2}{1+r^2-2r\cos(2\pi\theta)}$. We have $\phi_r(\theta + 1) = \phi_r(\theta) - 1$ and $\exp(-2\pi i\phi_r) = \frac{\exp(2\pi i\theta)-r}{1-r\exp(2\pi i\theta)}$. Let us observe that $U(z) = \frac{z-r}{1-rz}$ is a Blaschke factor. The function $\psi_r(\theta) = \phi_r(\theta) + \theta$ is periodic of period 1 and together with its derivative tends uniformly to 0 as r tends to 0. Indeed $\|\psi_r\|_\infty \le r/2$ and $\|\frac{d}{d\theta}\psi_r\|_\infty \le 2r/(1-r)$. A finite Blaschke product $U(z)$ could be used as well instead of a Blaschke factor and the continuous function ϕ_U would then be defined by $\exp(-2\pi i\phi_U(\theta)) = U(\exp(2\pi i\theta))$. We return to the construction of σ_r. Let $\Gamma_m \subset \mathbb{R}^2$, $m \in \mathbb{Z}$, be the curve in the plane defined by $x_2 = \phi_r(x_1) + m$ or equivalently by $x_2 = \psi_r(x_1) - x_1 + m$ and let $\mathcal{C} = \bigcup_{m\in\mathbb{Z}} \Gamma_m$ be the union of these disjoint curves. The curve Γ_m is contained in the strip W_m defined by $|x_1 + x_2 - m| \le r/2$. Since $0 \le r < 1$ these strips are pairwise disjoint. It implies the following:

Lemma 7. *If $x \in \Gamma_m, y \in \Gamma_l$ and $l \ne m$ we have $|x - y| \ge \frac{1-r}{\sqrt{2}}$.*

Definition 13. *The measure $\sigma_{(r,m)}$ on Γ_m is defined as the image of the Lebesgue measure dx_1 on the real axis by the map $x_1 \mapsto (x_1, \phi_r(x_1) + m)$.*

For any compactly supported continuous function f on $\mathbb{R}^2$ we have

$$\langle \sigma_{(r,m)}, f\rangle = \int_{\mathbb{R}} f(x_1, \phi_r(x_1) + m)dx_1. \tag{18}$$

Then $\sigma_{(r,m)}(x_1 + 1, x_2 - 1) = \sigma_{r,m)}(x_1, x_2)$.

Definition 14. *Let $\sigma_r = \sum_{-\infty}^{+\infty} \sigma_{(r,m)}$.*

From now on the index r is dropped: we write σ instead of σ_r. The value of $r \in (0, 1)$ does not play any role in what follows. The measure σ is $\mathbb{Z}^2$-periodic and is supported by $\mathcal{C}$. Let $\mathbf{B} \subset \mathbb{Z}^2$ be defined by $k_1 \ge 0, k_2 \ge 0$ or $k_1 \le 0, k_2 \le 0$. Then we have:

Lemma 8. *The Fourier coefficients $\widehat{\sigma}(k)$ of σ vanish if $k \notin \mathbf{B}$. Therefore σ is a Ahern measure.*

This is given by Theorem 8 which is applied to the inner function $U(z) = \frac{z-r}{1-rz}$. The measure σ_A is defined as the image measure of σ by an automorphism A of $\mathbb{R}^2$ and similarly σ_B is the image measure of σ by B. The matrices of these linear automorphisms are also denoted by A and B. Here is the definition of A and B which will be used below. Let us consider $A = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$ and $B = \begin{pmatrix} b_4 & -b_3 \\ -a_4 & a_3 \end{pmatrix}$ where

(i) $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0,$
(ii) $b_1 > 0, b_2 > 0,$
(ii) $b_3 < 0, b_4 < 0,$
(iv) $\det A = a_1 b_2 - a_2 b_1 = 1,$
(v) $\det B = a_3 b_4 - a_4 b_3 = 1.$

It implies $(A^*)^{-1} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ and $(B^*)^{-1} = \begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix}$.

The distributional Fourier transform of σ_A is $\widehat{\sigma} \circ A^*$. The support of $\widehat{\sigma} \circ A^*$ is contained in $(A^*)^{-1}(H) = H_A$. This $H_A \subset H$ is a narrower sector defined by $0 < c \leq x_2/x_1 \leq C$ where $C > c > 0$. Similarly The distributional Fourier transform of σ_B is $\widehat{\sigma} \circ B^*$ and is supported by $(B^*)^{-1}(H) = H_B \subset S$. This sector H_B is defined by $0 < c' \leq -x_2/x_1 \leq C'$. Let us observe that $H_A \cap H_B = \{0\}$. Since H_A and H_B are symmetric with respect to 0 this is Hörmander's sufficient condition.

Lemma 9. *The pairs (σ_A, H_A) and (σ_B, H_B) are two independent lighthouses.*

For instance if $\theta > 1$ and $a_1 = \theta, a_2 = \theta - 1, b_1 = \theta + 1, b_2 = \theta$, the sector H_A is delimited by the lines $x_2 = \frac{\theta}{\theta - 1} x_1$ and $x_2 = \frac{\theta + 1}{\theta} x_1$. The aperture of H_A is $O(\theta^{-2})$ and tends to 0 as θ tends to ∞. Similarly if $\theta' > 1$ and $a_3 = \theta' + 1, a_4 = \theta', b_3 = -\theta', b_4 = -\theta' + 1$, the sector H_B is delimited by the lines $x_2 = -\frac{\theta'}{\theta' + 1} x_1$ and $x_2 = -\frac{\theta' - 1}{\theta'} x_1$. The aperture of H_B is $O(\theta'^{-2})$ and tends to 0 as θ' tends to ∞.

We can assume that

$$(A^*)^{-1}(\mathbb{Z}^2) \cap (B^*)^{-1}(\mathbb{Z}^2) = \{0\}. \tag{19}$$

Indeed this is equivalent to the following condition: the vectors (a_1, b_1), (a_2, b_2), (a_3, b_3) and (a_4, b_4) are $\mathbb{Q}$-linearly independent. Let us observe that (19) is equivalent to $A(\mathbb{Z}^2) \cap B(\mathbb{Z}^2) = \{0\}$. In our example it amounts to say that $1, \lambda, \lambda'$ are $\mathbb{Q}$-linearly independent. This example suffices to our construction.

Lemma 10. *For any $k \in \mathbb{Z}$ and for any $l \in \mathbb{Z}$ the curve $A(\Gamma_k)$ is transverse to the curve $B(\Gamma_l)$.*

More precisely we have:

Lemma 11. *Each curve $A(\Gamma_k)$ is the graph of a decreasing function $g_{k,A}$ such that $g'_{k,A} \leq -\beta$ and each curve $B(\Gamma_k)$ is the graph of an increasing function $h_{k,B}$ such that $h'_{k,A} \geq \beta$. The positive number β only depends on $A, B,$ and r.*

Indeed the curve $A(\Gamma_k)$ admits a parametric representation given by $x_1 = b_2 t - b_1 \phi(t)$ and $x_2 = -a_2 t + a_1 \phi(t)$, $t \in \mathbb{R}$. Then x_1 is an increasing function of t and this function defines a diffeomorphism of $\mathbb{R}$. Next x_2 is a decreasing function of t. Finally x_2 is a decreasing function of x_1. Moreover $g_{k,A} : \mathbb{R} \mapsto \mathbb{R}$ is a diffeomorphism. The second assertion of is similar. Therefore the intersection $A(\Gamma_k) \cap B(\Gamma_l)$ is reduced to a point denoted by $\gamma_{k,l}$.

Lemma 12. *The set $\Lambda = \{\gamma_{k,l}; k \in \mathbb{Z}, l \in \mathbb{Z}\}$ is a Delone set.*

Before proving Lemma 12 let us observe that $\Lambda = \Lambda(A, B, r)$ depends on $A, B,$ and r. We first prove that Λ is uniformly discrete and then that Λ is relatively dense. If $x, y \in \Lambda$ are distinct two cases can occur. Either $x \in A(\Gamma_k)$, and $y \in A(\Gamma_l)$ with $k \neq l$ or $x \in B(\Gamma_k)$, and $y \in B(\Gamma_l)$ with $k \neq l$. In the first case $|x - y| \geq c_B > 0$ by Lemma 7. The second case is similar.

We now prove that Λ is relatively dense. Let L_m be the line defined by $x_1 + x_2 = m$. Let us write $\mathbf{e} = (1, 1), \mathbf{f} = (A^*)^{-1}(\mathbf{e}), \mathbf{g} = (B^*)^{-1}(\mathbf{e})$ and let us denote by M the lattice generated by $\mathbf{f}$ and $\mathbf{g}$. Let M^* be the dual lattice of M. Then M^* is also the set of all intersections $\widetilde{\gamma_{k,l}} = A(L_k) \cap B(L_l), k, l \in \mathbb{Z}$.

Lemma 13. *Let $r \in [0, 1)$. There exists a constant C depending only on the matrices A and B such that for any $k, l \in \mathbb{Z}$ we have*

$$|\gamma_{k,l} - \widetilde{\gamma_{k,l}}| \leq Cr. \tag{20}$$

Since M^* is a lattice Lemma 13 implies that Λ is relatively dense. Moreover Λ is a small perturbation of M^* if r is small. Let us prove Lemma 13. We already know that the curve Γ_m is contained in the strip W_m defined by $|x_1 + x_2 - m| \leq r/2$. Let us consider the intersection $K(k, l) = A(W_k) \cap B(W_l)$. The center of the parallelogram $K(k, l)$ is $\widetilde{\gamma_{k,l}}$. The diameter of this parallelogram $K(k, l)$ does not exceed Cr since the width of W_k does not exceed r. Therefore $K(k, l) \subset B(\widetilde{\gamma_{k,l}}, Cr)$ where $B(x, R)$ denotes the ball centered at x with radius R. Moreover $\gamma_{k,l} = A(\Gamma_k) \cap B(\Gamma_l)$ belongs to $K(k, l)$. It implies $|\gamma_{k,l} - \widetilde{\gamma_{k,l}}| \leq Cr$ and Lemma 13 is fully proved. One can observe that $K(k, l) = \widetilde{\gamma_{k,l}} + K(0, 0)$.

The pushforward measure $\sigma_A = \sigma \circ A^{-1}$ (the image measure of σ by A) is carried by $A(\mathcal{C})$ and is periodic with respect to the lattice $\Lambda_A = A(\mathbb{Z}^2)$. Similarly the pushforward measure $\sigma_B = \sigma \circ B^{-1}$ is carried by $B(\mathcal{C})$ and is periodic with respect to the lattice $\Lambda_B = B(\mathbb{Z}^2)$.

Definition 15. *The atomic measure μ is defined as the pointwise product between these two image measures σ_A and σ_B.*

Theorem 9. *This product measure $\mu = \sigma_A \, \sigma_B$ is a crystalline measure.*

Theorem 9 follows from Corollary 3 of Theorem 5. It can also be proved directly using the transversality between the curves $A(\Gamma_k)$ and $B(\Gamma_l)$. The support of μ is the Delone set $\Lambda = \{\gamma_{k,l}; k \in \mathbb{Z}, l \in \mathbb{Z}\}$ and we have $\mu = \sum c(k, l)\delta_{\gamma_{k,l}}$. If $\gamma_{k,l} = (\alpha_{k,l}, \beta_{k,l})$ then

$$c(k, l) = (h'_{l,B}(\alpha_{k,l}) - g'_{k,A}(\alpha_{k,l}))^{-1} > 0 \tag{21}$$

where g' denotes the derivative of g and h' denotes the derivative of h.

Let us observe that one could use the function ϕ_r for defining the curves $A(\Gamma_k)$ and a different function ϕ_s for defining the curves $B(\Gamma_l)$. One could use as well the phases of two arbitrary Blaschke products. Then the spectrum of μ is always contained in the locally finite set $H_A \cap \Lambda_{A^*} + H_B \cap \Lambda_{B^*}$ which does not depend on the two inner functions which are used in the construction. The measure μ and its support Λ obviously depend on these inner functions.

6. The second construction

In this second construction a lighthouse (w, T) is constructed directly without using an Ahern measure [9]. Then Theorem 5 is used again to construct a crystalline measure. Here are the details of these two steps. Let T be the double cone defined by

$$T = \{k \in \mathbb{Z}^2; |k_2| \leq |k_1|\}. \tag{22}$$

The core of the second construction is the following lemma:

Lemma 14. *There exists a lighthouse (w, T) on $\mathbb{T}^2$ enjoying the following properties:*

(a) *w is a probability measure supported by the disjoint union $\Gamma \subset \mathbb{T}^2$ of two closed smooth curves.*

(b) *w is absolutely continuous with respect to the arc length measure on Γ.*

Before detailing this construction let us show that it immediately yields one dimensional and two dimensional crystalline measures. Let $\alpha \in (0, \pi/4)$ and let L_α be the line defined by the parametric representation $x_1 = t\cos\alpha, x_2 = t\sin\alpha,\ t \in \mathbb{R}$. The Lebesgue measure on L_α defines a two dimensional Radon measure λ_α by $\langle \lambda_\alpha, g\rangle = \int_{\mathbb{R}} g(t\cos\alpha, t\sin\alpha)\, dt$ for any compactly supported continuous function g. Then the distributional Fourier transform of λ_α is $\lambda_{(\alpha+\pi/2)}$. Therefore $(\lambda_\alpha, L_{(\alpha+\pi/2)})$ is a trivial lighthouse. The lighthouse (w, T) on $\mathbb{T}^2$ can be viewed as a lighthouse (τ, T) on $\mathbb{R}^2$, τ being the $\mathbb{Z}^2$ periodic version of w. One considers the pointwise product between τ and λ_α.

Lemma 15. *The pointwise product $\rho = \tau\lambda_\alpha$ is a one dimensional crystalline measure on the line L_α.*

The definition of this pointwise product is given by Theorem 5. It can also be verified directly using the properties of Γ. Let us take for granted that ρ is an atomic measure supported by a locally finite set. This will be obvious once Γ is defined. The two dimensional distributional Fourier transform of ρ is the convolution product $\omega * \lambda_{(\alpha+\pi/2)}$ between the atomic measure $\omega = \sum_{k\in\mathbb{Z}^2} \widehat{w}(k)\delta_k$ and $\lambda_{(\alpha+\pi/2)}$. The one dimensional Fourier transform of ρ is defined using the parametric representation of L_α. This one dimensional Fourier transform of ρ is the atomic measure $\sum_{k\in\mathbb{Z}^2} \widehat{w}(k)\delta_{(k_1\cos\alpha + k_2\sin\alpha)}$ which is the restriction to L_α of the two dimensional Fourier transform $\widehat{\rho}$. The support of $\sum_{k\in\mathbb{Z}^2} \widehat{w}(k)\delta_{(k_1\cos\alpha+k_2\sin\alpha)}$ is locally finite since the line $L_{(\alpha+\pi/2)}$ is transverse to T. This ends the proof. The same scheme yields all the crystalline measures which were named “curved model sets” in [12].

We now follow [9] and construct some two dimensional crystalline measures. This time we start from a positive one dimensional crystalline measure ν and consider the tensor product $\nu_2 = \nu \otimes dx_2$. The one dimensional Fourier transform of ν is $\widehat{\nu} = \sum_{y\in F} a(y)\delta_y$. Then the two dimensional Fourier transform of ν_2 is $\widehat{\nu} = \sum_{y\in F} a(y)\delta_{(y,0)}$. We rotate ν_2 around 0 with an angle $\alpha \in (\pi/4, 3\pi/4)$ and obtain the measure $\nu_{2,\alpha}$ which is a trivial lighthouse. As above the sharp lighthouse (w, T) on $\mathbb{T}^2$ can be viewed as a lighthouse (τ, T) on $\mathbb{R}^2$, τ being the $\mathbb{Z}^2$ periodic version of w. With these notations we have:

Theorem 10. *The pointwise product $\nu_{2,\alpha}\tau$ is a two dimensional crystalline measure.*

Once more this is given by Corollary 3 of Theorem 5.

Following the recipe given in [9] we now prove Lemma 14 and construct w. Let $r \in [0,1]$ and let θ_r be the 1-periodic continuous function defined on $\mathbb{T}$ by $\cos(2\pi\theta_r(x)) = r\cos(2\pi x)$ and $0 \le \theta_r(x) \le 1/2$. We have $\theta_0(x) = 1/4$ identically and $\theta_1(x) = |x|$ if $-1/2 \le x \le 1/2$. The function θ_r is analytic if $0 \le r < 1$. The derivative of $\theta_r(x)$ is

$$\frac{d\,\theta_r}{dx} = \frac{r\sin(2\pi x)}{\sqrt{1 - r^2\cos^2(2\pi x)}} \tag{23}$$

We have $\theta_r(1/2 - x) + \theta_r(x) = 1/2$ and $\|\theta_r - 1/4\|_\infty < r/4$ if $0 \le r < 1$. The following lemma is seminal in our construction:

Lemma 16. *The* $1-$*periodic function* $\cos(2\pi m\theta_r(x)), m \in \mathbb{N},$ *is a trigonometric polynomial. More precisely we have*

$$\cos(2\pi m\theta_r(x)) = \sum_{k=0}^{m} \alpha_r(k,m)\cos(2\pi kx). \tag{24}$$

The proof is elementary [9]. We extend $\alpha_r(k,m)$ to $\mathbb{N}^2$ by setting $\alpha_r(k,m) = 0$ if $k \notin [0,m]$. A first curve $\Gamma^+ \subset \mathbb{T}^2$ is defined by $x_1 = \theta_r(x_2)$ and a second one $\Gamma^- \subset \mathbb{T}^2$ is defined by $x_1 = -\theta_r(x_2)$. Let w_+ be the measure on $\mathbb{T}^2$ which is supported by Γ^+ and is defined by the following property: for any compactly supported continuous function g on $\mathbb{R}^2$ we have $\langle w_+, g\rangle = \int_{\Gamma^+} g(x)\,dx_2 = \int_{\mathbb{R}} g(\theta_r(x_2), x_2)dx_2$. Then we have

Lemma 17. *The Fourier coefficients of* w_+ *are*

$$\widehat{w}_+(k_1,k_2) = \int_0^1 \exp(-2\pi i k_1\theta_r(x_2))\exp(-2\pi i k_2 x_2)\,dx_2. \tag{25}$$

In a similar way we consider the measure w_- on $\mathbb{T}^2$ which is supported by Γ^- and is defined by the following property: for any compactly supported continuous function g on $\mathbb{R}^2$ we have $\langle w_-, g\rangle = \int_{\Gamma^-} g(x)\,dx_2 = \int_{\mathbb{R}} g(-\theta_r(x_2), x_2)dx_2$.

Lemma 18. *The Fourier coefficients of* w_- *are*

$$\widehat{w}_-(k_1,k_2) = \int_0^1 \exp(2\pi i k_1\theta_r(x_2))\exp(-2\pi i k_2 x_2)\,dx_2. \tag{26}$$

Finally we consider $w = w_+ + w_-$. Then the measure w is supported by $\Gamma = \Gamma^+ \cup \Gamma^-$. The Fourier coefficients of w are

$$\widehat{w}(k_1,k_2) = 2\int_0^1 \cos(2\pi k_1\theta_r(x_2))\exp(-2\pi i k_2 x_2)\,dx_2. \tag{27}$$

But $\cos(2\pi k\theta_r(x)) = \sum_0^k \alpha_r(k,m)\cos(2\pi kx)$. This yields

$$\widehat{w}(k_1,k_2) = \alpha_r(|k_2|,|k_1|). \tag{28}$$

Lemma 19. *The pair* (w, T) *is a lighthouse on* $\mathbb{T}^2$*.*

Viewed as two $\mathbb{Z}^2-$periodic measures on $\mathbb{R}^2$ the measures $w_\pm$ and w are denoted by $\tau_\pm$ and τ The support of τ_+ is the union $\mathcal{C}_+$ of the pairwise disjoint curves $\mathcal{C}_l^+, l \in \mathbb{Z}$, defined by the equations $x_1 = \theta_r(x_2) + l$. Similarly the support of τ_- is the union $\mathcal{C}_-$ of the pairwise disjoint curves $\mathcal{C}_l^-, l \in \mathbb{Z}$, defined by the equations $x_1 = -\theta_r(x_2) + l$. We set $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$. Let us summarize our discussion.

Lemma 20. *The $\mathbb{Z}^2-$periodic measure τ is supported by $\mathcal{C}$ and the distributional Fourier transform of τ is an atomic measure ρ supported by $T \cap \mathbb{Z}^2$. Therefore (τ, T) is a lighthouse.*

7. Exotic Ahern measures

In Theorem 8 the inner function f is assumed to be smooth. What happens if this condition is not satisfied? We show here that some singular inner functions yield beautiful Ahern measures. However these exotic measures cannot be used to construct crystalline measures. It is interesting to analyze this failure. In our first example $\alpha > 0$ is a constant and the singular inner function $\mathbf{g}(z) = \exp(-\alpha\frac{1-z}{1+z})$ is used. Then the proof of Theorem 8 can be repeated and an Ahern measure $\omega_\mathbf{g}$ on $\mathbb{T}^2$ is constructed. We now compute $\omega_\mathbf{g}$. We set $z = \exp(2\pi i\theta)$. The phase $\phi(\theta)$ of $\mathbf{g}(2\pi i\theta)$ is $\phi(\theta) = \alpha\frac{\tan(\pi\theta)}{2\pi}$. Let $\Gamma_\mathbf{g} \subset \mathbb{T}^2$ be the graph of $-\phi : \mathbb{T} \mapsto \mathbb{T}$. Then $\Gamma_\mathbf{g}$ is an open curve in $\mathbb{T}^2$ with an infinite length. The closure F of $\Gamma_\mathbf{g}$ is the union between $\Gamma_\mathbf{g}$ and the vertical line defined by $\theta_1 = 1/2$. Therefore F has a zero Haar measure. An Ahern measure $\omega_\mathbf{g}$ supported by $\Gamma_\mathbf{g}$ is constructed as follows. One defines the probability measure $\omega_\mathbf{g}$ as the image of the Haar measure on $\mathbb{T}$ by the mapping $\theta_1 \mapsto (\theta_1, -\phi(\theta_1))$. Then $\omega_\mathbf{g}$ is absolutely continuous with respect to the arc length measure ds on $\Gamma_\mathbf{g}$. The density of $\omega_\mathbf{g}$ with respect to ds is a smooth function of the arc length s and is $O(s^{-2})$ at infinity. Finally the proof of Theorem 8 can be repeated in this context and implies that $\omega_\mathbf{g}$ is an Ahern measure on $\mathbb{T}^2$. We now check that this Ahern measure $\omega_\mathbf{g}$ cannot be used to construct a crystalline measure. For $m \in \mathbb{Z}$ we let $\Gamma_m \subset \mathbb{R}^2$ be the graph of the function $-\phi + m : \mathbb{R} \mapsto \mathbb{R}$. Let σ_m be the image of the Lebesgue measure on the horizontal axis by the map $x_1 \mapsto (x_1, -\phi(x_1) + m)$. We consider $\mathcal{C} = \cup_m \Gamma_m$ which is the support of the measure $\sigma = \sum_m \sigma_m$. This set $\mathcal{C}$ is not closed in $\mathbb{R}^2$. The closure of $\mathcal{C}$ is the union between $\mathcal{C}$ and the vertical lines defined by $x_1 = k + 1/2,\ k \in \mathbb{Z}$. Next we consider the line $L = \{x \in \mathbb{R}^2; x_1 = t, x_2 = \sqrt{2}t,\ t \in \mathbb{R}\}$. Let τ_L be the Lebesgue measure on L. The pointwise product $\sigma\tau_L$ is not a crystalline measure on L. Indeed the intersection $\Lambda = \mathcal{C} \cap L$ is not a closed discrete set since every point $x \in L$ of the form $x = (k + 1/2, \sqrt{2}(k + 1/2)), k \in \mathbb{Z}$, is an accumulation point of Λ. We can generalize this example with $\mathbf{g}(z; \alpha, \zeta) = \exp(-\alpha\frac{\zeta - z}{\zeta + z})$ with $|\zeta| = 1$ and finally with a finite product $\mathbf{G}(z) = \mathbf{g}(z; \alpha_1, \zeta_1) \ldots \mathbf{g}(z; \alpha_N, \zeta_N)$. Then the phase of $\mathbf{G}(z)$ is $\phi_\mathbf{G}(\theta) = \sum_1^N \frac{\alpha_j}{2\pi}\tan(\pi(\theta - \theta_j))$ if $\zeta_j = \exp(2\pi i\theta_j)$. Here again $\Gamma_\mathbf{G} \subset \mathbb{T}^2$ is the graph of $-\phi_\mathbf{G} : \mathbb{T} \mapsto \mathbb{T}$. Then $\Gamma_\mathbf{G}$ is an open curve in $\mathbb{T}^2$ with an infinite length.

In this second example we address Theorem 7 when the inner function f is not smooth. As above $\mathbf{g}(z) = \exp(-\frac{1-z}{1+z})$. The inner function is given by $\mathbf{f}(z, \zeta) = \mathbf{g}(z)\mathbf{g}(\zeta)$ on $\mathbb{T}^2$. Let us then consider $\mu_\mathbf{f} = \Re(\frac{1+\mathbf{f}}{1-\mathbf{f}})$. This positive measure is supported by the set

$\Gamma = \bigcup_{-\infty}^{+\infty} \Gamma_m$ where Γ_m is the curve defined by $\Gamma_m = \{\theta \in \mathbb{T}^2; \tan(\pi\theta_1) + \tan(\pi\theta_2) = 2m\pi,\ m \in \mathbb{Z}\}$. Then each Γ_m is the graph of a smooth function $-\phi_m$. We have $\phi_0(t) = t$. The derivative of ϕ_m is

$$\phi'_m(t) = \frac{2}{2 + 4\pi^2 m^2 + 4\pi m \sin(2\pi t) + 4\pi^2 m^2 \cos(2\pi t)} = P_r(t - \alpha)$$

where P_r is the Poisson kernel, $r = \frac{2\pi m}{\sqrt{4+4\pi^2 m^2}}$, and $\tan\alpha = 1/\pi m$. We have $\phi_m(\pm 1/2) = \pm 1/2$. Moreover $\phi'_m(\pm 1/2) = 1$. Therefore all the curves Γ_m are tangent at $(1/2, -1/2)$ and at $(-1/2, 1/2)$. The proof of Theorem 7 can be repeated here and we obtain an Ahern measure $\omega_{\mathbf{g},\mathbf{g}}$ supported by Γ. This Ahern measure cannot be used to construct crystalline measures. To prove this remark we lift each Γ_m from $\mathbb{T}^2$ to $\mathbb{R}^2$ and denote by $\mathcal{C}$ the union of these lifted curves. All the curves Γ_m meet at $(1/2, -1/2)$ where they are tangent. As in our first example this prevents us from using the measure $\mu_{\mathbf{f}}$ to construct crystalline measures.

8. Counter examples

Theorem 5 is not valid if $u_1, \dots, u_n$ are signed measures. Here is a counter example in four dimensions with $n = 2$. We have not been able to construct a two dimensional example. We begin with the building blocks of our counter example. The double cones $S \subset \mathbb{R}^2$ and $T \subset \mathbb{R}^2$ are defined by $S = \{x; |x_2| \le |x_1|/3\}$ and $S = \{x; |x_1| \le |x_2|/3\}$. They are independent. We construct two functions $u \in L^1(\mathbb{R}^2)$ and $v \in L^1(\mathbb{R}^2)$ which have the required spectral properties. However their pointwise product is not a measure. These functions are the building blocks to construct the two lighthouses which are the counter examples. Let ϕ be a non trivial even function in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ such that $\widehat{\phi}$ is a non negative function supported by the unit disc. The function $u \in L^1(\mathbb{R}^2)$ is defined by

$$u(x) = \sum_1^\infty k^{-2} 4^k \exp(2\pi i 7^k x_1) \phi(2^k x) \tag{29}$$

and v is defined by $v(x_1, x_2) = u(x_2, x_1)$.

Lemma 21. *The two functions u and v are continuous on $\mathbb{R}^2 \setminus \{0\}$ and have a fast decay at infinity. They belong to $L^1(\mathbb{R}^2)$ The Fourier transform of u is supported by S and the Fourier transform of v is supported by T. The pointwise product uv is a tempered distribution which is not a measure.*

The Fourier transform of $\phi(2^k x)$ is supported by the ball centered at 0 with radius 2^k. Therefore the Fourier transform of $\exp(2\pi i 7^k x_1)\phi(2^k x)$ is carried by the disc centered at $(7^k, 0)$ with radius 2^k. This disc is contained in S. It implies that the Fourier transform of u is supported by S. The same argument applies to v and T. We now prove that the distribution $w = uv$ is not a measure. We argue by contradiction. If w was a measure it would be a bounded measure since both u and v have a fast decay at infinity. To reach a contradiction we prove now that the Fourier transform of w is unbounded. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We have $\widehat{w} = \widehat{u} * \widehat{v} = \sum_{k,l} k^{-2} l^{-2} J_{k,l}$ where

$$J_{k,l}(x) = \int \widehat{\phi}\left(\frac{x - 7^k e_1 - y}{2^k}\right) \widehat{\phi}\left(\frac{y - 7^l e_2}{2^l}\right) dy. \tag{30}$$

For any $k, l \in \mathbb{N}$ we have $J_{k,l}(x) \geq 0$ since $\widehat{\phi} \geq 0$. This implies $\|\widehat{w}\|_\infty \geq k^{-4}\|J_{k,k}\|_\infty \geq k^{-4}J_{k,k}(7^k, 7^k) = k^{-4}4^k\|\phi\|_2^2$ which ends the proof.

It remains to construct two "signed lighthouses measures" μ and ν whose pointwise product is not a measure. A "signed lighthouse" is defined by (i), (ii), and (iii) in Definition 9 but we do not impose that μ be positive. Our first guess is to use u and v. But the functions u and v do not satisfy (ii). We now address this issue. For $(x_1, \dots, x_4) \in \mathbb{R}^4$ we set $x' = (x_1, x_2)$ and $x" = (x_3, x_4)$ and consider the measures $\mu = dx_1 \otimes \delta(x_2) \otimes u(x")$ and $\nu = \delta(x_1) \otimes dx_2 \otimes v(x")$. These measures are two independent lighthouses measures. But the pointwise product between μ and ν is $\delta(x_1) \otimes \delta(x_2) \otimes uv(x")$ which is not a measure.

9. Direct proofs of Theorems 4 and 5

A pedestrian proof of Theorem 4 is detailed for $n = 2$. We set $u_1 = u, u_2 = v, S_1 = S$, and $S_2 = T$. Let $\|u\|$ (resp. $\|v\|$) the total mass of u (resp. the total mass of v). Let $a(k), k \in \mathbb{Z}^n$, be the Fourier coefficients of u and $b(k), k \in \mathbb{Z}^n$, be the Fourier coefficients of v. Let $G_N(x)$ be the Fejer kernel in two variables. We consider $u_N = G_N * u$ and $v_N = G_N * v$. Then u_N and v_N are two non negative trigonometric polynomials. We have $\|u_N\|_1 \leq \|u\|, \|v_N\|_1 \leq \|v\|$ and the Fourier coefficients $a_N(k)$ and $b_N(k), k \in \mathbb{T}^n$, are supported by S and T and tend to $a(k)$ and $b(k)$ as N tends to infinity. Then the pointwise product $u_N v_N$ is obviously a non negative trigonometric polynomial.

Lemma 22. *We have*

$$I_N = \int_{\mathbb{T}^n} u_N v_N \, dx = \|u\|\|v\|. \tag{31}$$

Indeed this integral is given by $I_N = \sum_{k+l=0} a_N(k)b_N(l)$. But $k \in S, l \in T$, and $k + l = 0$ imply $k = l = 0$ which ends the proof of (31). To define uv we need to show that the pointwise products $u_N v_N$ converge in the distributional sense. Since $u_N v_n$ is a bounded sequence in $L^1(\mathbb{T}^n)$ it suffices to show that each of the Fourier coefficients $c_N(k)$ of $u_N v_N$ has a limit as N tends to infinity. Indeed $c_N(m) = \sum_{k+l=m} a_N(k)b_N(l)$. But given m the set E_m of pairs $(k, l) \in S \times T$ such that $k + l = m$ is a finite set independent from N. Therefore $c_N(m) = \sum_{\{k+l=m,(k,l)\in E_m\}} a_N(k)b_N(l)$ tends to $c(m) = \sum_{\{k+l=m,(k,l)\in E_m\}} a(k)b(l)$ as N tends to infinity. The same argument shows that the definition of the product uv does not depend on the choice of the Fejer kernel. Indeed we have:

Lemma 23. *Let u and v satisfy the hypotheses of Theorem 4. Let u_m be a sequence of non negative trigonometric polynomials such that (a) $\int u_m = 1$, (b) the Fourier coefficients of u_m vanish outside S, and (c) $u_m \rightharpoonup u$ as $m \to \infty$. Let us assume that v_m satisfies the same conditions with respect to v. Then the pointwise products $u_m v_m$ tend to uv in the distributional sense as $m \to \infty$.*

The proof is a copy of the preceding one.

We now prove Theorem 5. Let ϕ be a positive even function in the Schwartz class such that $\widehat{\phi}$ is supported by the unit ball denoted by B. To prove that uv is a positive measure it suffices to replace u and v by $U = \phi u$ and $V = \phi v$ and to prove that the pointwise product UV is a positive measure. Now U and V are bounded Radon measures. Let $\widehat{u}$ be the distributional Fourier transform of u. Then the Fourier transform of U is $\widehat{U}(y) = \widehat{u} * \widehat{\phi}$ and similarly the Fourier transform of V is $\widehat{V}(y) = \widehat{v} * \widehat{\phi}$. These Fourier transforms are supported by $S+B$ and $T+B$ where S and T are two independent double cones. Let $G(x)$ be a non negative Schwartz function such that $\int G = 1$. Let us assume that the Fourier transform of G is supported by the unit ball B. Let $G_N(x) = N^n G(Nx)$, $U_N = U * G_N$ and $V_N = V * G_N$. Then U_N and V_N are non negative Schwartz functions. Their Fourier transforms are given by

$$\widehat{U}_N(y) = \widehat{G}(y/N) \int \widehat{U}(z)\widehat{\phi}(y-z)\, dz \tag{32}$$

and

$$\widehat{V}_N(y) = \widehat{G}(y/N) \int \widehat{V}(z)\widehat{\phi}(y-z)\, dz. \tag{33}$$

Lemma 24. *There exists a positive constant C such that for any $N \geq 1$ we have $\int_{\mathbb{R}^n} U_N V_N \, dx \leq C$.*

Indeed $I_N = \int_{\mathbb{R}^n} U_N V_N = \int_{\mathbb{R}^n} \widehat{U}_N \overline{\widehat{V}_N}$. We now use (32) and (33) to expand $\widehat{U}_N$ and $\widehat{V}_N$. We obtain

$$I_N = \int_{\mathbb{R}^{3n}} |\widehat{G}(y/N)|^2 \widehat{U}(z)\widehat{\phi}(y-z)\overline{\widehat{V}}(z')\widehat{\phi}(y-z')\, dydzdz'. \tag{34}$$

A product $\widehat{\phi}(y-z)\widehat{\phi}(y-z')$ vanishes if $|y-z| \geq 1$ or $|y-z'| \geq 1$. Therefore the domain of integration of the integral in (34) is contained in

$$A = \{(y,z,z'); z \in S+B, z' \in T+B, |y-z| \leq 1, |y-z'| \leq 1\}.$$

But Lemma 2 implies that A is a compact set. Then Lemma 24 follows immediately from the trivial estimates $|\widehat{U}| \leq \|U\|$ and $|\widehat{V}| \leq \|V\|$. It remain to prove that $U_N V_N \rightharpoonup UV$ as $N \to \infty$. The proof is similar to the preceding one. We are led to showing that $\widehat{U}_N * \widehat{V}_N \rightharpoonup \widehat{U} * \widehat{V}$ as $N \to \infty$ or $\int (\widehat{U}_N * \widehat{V}_N)(y) f(y)\, dy \to \int (\widehat{U} * \widehat{V})(y) f(y)\, dy$ for any compactly supported continuous function f. The support of f is contained in a ball centered at 0 with radius R. Let us observe that $\widehat{U}_N * \widehat{V}_N(y)$ is given by the integral

$$J(y) = \int_{\mathbb{R}^{3n}} \widehat{G}((y-z)/N)\widehat{U}(z')\widehat{\phi}(y-z-z')\widehat{G}(z/N)\widehat{V}(z'')\widehat{\phi}(z-z'')\, dzdz'dz''.$$

Here again $|y| \leq R, |y-z-z'| \leq 1, |z-z''| \leq 1, z' \in S+B, z'' \in T+B$, imply that these arguments belongs to a bounded set K which does not depend on N. The conclusion follows immediately.

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