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Skrifter nr. 2-2021

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THE ROYAL NORWEGIAN SOCIETY OF SCIENCES AND LETTERS

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ISBN 978-82-93175-68-1 Trykk
ISBN 978-82-93175-69-8 Web
ISSN 0368-6310

Layout and print: Skipnes Kommunikasjon AS

Skipnes.no Svanemerket trykksak, 2041 0731

Transactions of The Royal Norwegian Society of Sciences and Letters
(Trans. R. Norw. Soc. Sci. Lett. 2021(2), 5-30)

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Yves François Meyer

Abstract. A crystalline measure is an atomic measure μ which is supported by a locally finite set and whose distributional Fourier transform is also supported by a locally finite set. In this note linear recurrence relations on lattices are used to construct crystalline measures supported by uniformly discrete sets. Previous results by Pavel Kurasov and Peter Sarnak on the one hand and by Alexander Olevskii and Alexander Ulanovskii on the other hand are recovered on the way.

1. Introduction

An atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if the three following conditions are satisfied: (i) μ is supported by a locally finite set, (ii) μ is a tempered distribution, and (iii) the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set. This is equivalent to a generalized Poisson summation formula, as it is shown in Section 3. In the late fifties crystalline measures were studied by André-Paul Guinand[1], Jean-Pierre Kahane, and Szolem Mandelbrojt [4] who were motivated by the relation between (a) the functional equation satisfied by the Riemann zeta function and (b) the standard Poisson formula. Here is the issue. Let $\mu = \sum_{k=-\infty}^{\infty} a_k\delta_{\lambda_k}$ be an even crystalline measure which does not charge 0. Then we set $\phi(s) = \sum_{k=1}^{\infty} a_k\lambda_k^{-s}$. Since μ is a crystalline measure we have $\widehat{\mu} = \sum_{k=-\infty}^{\infty} b_k\delta_{\omega_k}$. Let us assume that $\widehat{\mu}$ does not charge 0 and let $\psi(s) = \sum_{k=1}^{\infty} b_k\omega_k^{-s}$.

Key words and phrases. Poisson summation formula, crystalline measure.

The referees were extremely helpful. One of them provided us with Figures 1 and 2. This work was supported by a grant from the Simons Foundation (601950, YM).

[1] A.P. Guinand (1912–1987) was born in Australia. His education was supported by scholarships. One of the supervisors of his thesis was Hardy. He served in the Royal Canadian Air Force from 1940 to 1945. Guinand's research included work in number theory (particularly prime numbers and the Riemann hypothesis), as well as generalizations of the Fourier transform.

Then $\phi(s)$ and $\psi(s)$ are two entire functions which are connected by the functional equation $\pi^{-s/2}\Gamma(\frac{s}{2})\phi(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\psi(1-s)$. Here Γ is the Euler gamma function. In [4] Kahane and Mandelbrojt wanted to know if the converse implication is true. Does the validity of the functional equation imply that the distributional Fourier transform of $\mu = \sum_{k=-\infty}^{\infty} a_k\delta_{\lambda_k}$ is $\sum_{k=-\infty}^{\infty} b_k\delta_{\omega_k}$? This line of research can be traced back to Hans Ludwig Hamburger [2]. When Kahane and Mandelbrojt wrote their paper it was debated whether or not a crystalline measure is necessarily a generalized Dirac comb. But this same year, Guinand discovered a revolutionary crystalline measure. In his seminal work [1] Guinand constructed an explicit atomic measure μ supported by the set $\Lambda = \{\pm\sqrt{k+1/9},\, k \in \mathbb{N}\cup\{0\}\}$ and claimed that $\widehat{\mu} = \mu$.[2] This beautiful example of a crystalline measure is rooted in Guinand's work on number theory. Unfortunately, the proof given by Guinand in [1] was incomplete, as was noticed by Olevskii in [8]. The fascinating problems raised by Guinand were forgotten for more than fifty years. Fortunately in 2015 Nir Lev and Alexander Olevskii gave a new life to Guinands's work and constructed a crystalline measure which is not a generalized Dirac comb [12]. A few months later Guinand's claims were proved [14]. Very likely the revival of Guinand's work is due to the discovery of quasi-crystals by Dan Shechtman (1982). A quasi-crystal is modeled by an atomic measure μ enjoying two properties at least: (i) μ is a weighted sum of Dirac measures on a uniformly discrete set Λ and (ii) the diffraction image of μ is also an atomic measure. One often adds a fivefold symmetry requirement. The diffraction image of μ is a renormalized version of $|\widehat{\mu}|^2$. One could believe that a quasi-crystal is a crystalline measure. It is not the case, as it was proved by Lev and Olevskii. This issue is related to the problem of stable interpolation, as it is noticed in Section 9.

A set Λ is uniformly discrete if the set M of distances between two different elements of Λ has a positive lower bound. A uniformly discrete set of real numbers has a finite upper density. In Guinand's example as well as in the construction by Lev and Olevskii the density of the support of the crystalline measure is infinite. That explains why the construction of a non trivial crystalline measure whose support is a uniformly discrete set is a spectacular result. Pavel Kurasov achieved this feat. Quantum graphs are seminal in this beautiful discovery. Recently Pavel Kurasov and Peter Sarnak replaced quantum graphs by a clever argument using some special polynomials (named *stable polynomials*) and Cauchy's residue theorem [5]. *Stable polynomials* originate from quantum graphs but can be defined independently. Soon after Alexander Olevskii and Alexander Ulanovskii discovered a new family of crystalline measures supported by uniformly discrete sets [16].

Four constructions of crystalline measures supported by uniformly discrete sets are described in this note. The third construction is the more general and is detailed in Theorem 5. The crystalline measures obtained from the other constructions can also be deduced from Theorem 5. Our first construction uses linear recurrence relations on lattices. These recurrence relations are studied in Section 2. Some notations and the definition of crystalline measures are presented in Section 3. The results obtained in Section 2 are used in Section 4 to construct a non trivial crystalline measure μ supported

[2] In Guinand's work as well as in [4] the Fourier transform of f is $\widehat{f}(y) = \int_{\mathbb{R}} \exp(-2\pi ixy)f(x)\,dx$.

by a uniformly discrete set. This short proof does not unveil the geometric properties of μ and does not provide us with a crystalline measure of the form $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ where Λ is a uniformly discrete set. Curved model sets [12] are needed to prove the fine results obtained by Kurasov and Sarnak on the one hand, and by Olevskii and Ulanovskii on the other hand. This second construction of crystalline measures is detailed in Section 5. Our third construction (Theorem 5) relies on the theory of mean-periodic functions. The relation between crystalline measures and mean-periodic functions is deep and seminal as is shown in Section 8 and in a spectacular paper by Olevskii and Ulanovskii [17]. Finally crystalline measures are obtained by almost periodic perturbations of lattices in Section 9. In Section 10 we return to [4] and we unveil the general Dirichlet series which are associated to crystalline measures. Our next goal is to extend the present work to the construction of crystalline measures on $\mathbb{R}^n$.

2. Recurrence relations

The Fibonacci sequence is defined by the recurrence relation:

$$c_{k+1} = c_k + c_{k-1},\ k \in \mathbb{N}, \tag{1}$$

with the initial conditions $c_0 = 0, c_1 = 1$. The Fibonacci sequence has an exponential growth at infinity. If one started with the initial conditions $c_0 = 1, c_1 = 0$, one would obtain the same Fibonacci sequence shifted by 1. Our construction of crystalline measures relies on linear recurrence relations indexed by $k \in \mathbb{Z}^m$ instead of the usual $k \in \mathbb{N}$. As an example let us compute the full Fibonacci sequence, defined as the Fibonacci sequence indexed by $k \in \mathbb{Z}$. We start from $c_0 = 0, c_1 = 1$. We obtain $c_{-1} = 1, c_{-2} = -1, c_{-3} = 2, c_{-4} = -3$ and finally $c_{-k} = (-1)^{k+1}c_k,\ k \in \mathbb{N}$.

The exponential growth of the Fibonacci sequence sharply contrasts with what happens to the solutions of the recurrence relation:

$$c_{k+1} - 2rc_k + c_{k-1} = 0,\ k \in \mathbb{Z}, \tag{2}$$

when $r \in (-1, 1)$. Any solution of (2) is an almost periodic function of k. Indeed the characteristic polynomial of (2) is $P(z) = z^2 - 2rz + 1$. The roots of P are $z = \exp(\pm i\phi)$ where $\phi \in (0, \pi)$ is defined by $\cos\phi = r$. The solutions of (2) are given by $c_k = a \exp(ik\phi) + b \exp(-ik\phi)$ where a and b are two constants. The same property holds for the equation

$$c_{k+3} - (1/2)c_{k+2} - (1/2)c_{k+1} + c_k = 0. \tag{3}$$

The characteristic polynomial of (3) is $P(z) = z^3 - (1/2)z^2 - (1/2)z + 1$. The roots of P are $z = -1$ and $z = 3/4 \pm i\sqrt{7}/4$. These three roots z_1, z_2, z_3, satisfy $|z| = 1$. It implies that any sequence c_k, $k \in \mathbb{Z}$, fulfilling (3) is almost periodic. It is the sequence of Fourier–Stieltjes coefficients of an atomic measure supported by $\{z_1, z_2, z_3\} \subset \mathbf{T}$. The circle group $\mathbf{T}$ is defined by $\{z \in \mathbb{C}; |z| = 1\}$.

A last example is given by the recurrence relation

$$c_{k+4} + 2c_{k+3} - 2c_{k+2} + 2c_{k+1} + c_k = 0 \tag{4}$$

where $k \in \mathbb{Z}$. Any bounded solution $c_k \in l^\infty(\mathbb{Z})$ of (4) is an almost periodic sequence. Indeed the roots of the characteristic polynomial of (4) are $\alpha \pm i\sqrt{\alpha}$ and $\phi \pm \sqrt{\phi}$ where

$\alpha = \frac{-1+\sqrt{5}}{2}$ and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Let us observe that $\theta = \phi + \sqrt{\phi}$ is a Salem number. A Salem number is an algebraic integer whose conjugates (θ being excepted) all belong to the closed unit disc, one of these conjugates at least belonging to the unit circle. In our example we have $|\alpha \pm i\sqrt{\alpha}| = 1$ and $|\phi - \sqrt{\phi}| < 1$. The bounded solutions of (4) are given by $c_k = a\,(\alpha + i\sqrt{\alpha})^k + b\,(\alpha - i\sqrt{\alpha})^k$, $k \in \mathbb{Z}$, where a and b are two constants. The dimension of the vector space of all solutions of (4) is 4 but the dimension of the vector space of bounded solutions is only 2. This phenomena will be met again in this note.

Our first proof of the existence of crystalline measures supported by uniformly discrete sets relies on the properties of bounded solutions to the linear recurrence relation defined on $\mathbb{Z}^2$ by:

$$c(m+1,n+1) = (1/2)c(m+1,n) + (1/2)c(m,n+1) - c(m,n) \tag{5}$$

where $(m,n) \in \mathbb{Z}^2$. One could use other recurrence relations. For instance one could replace the coefficient $1/2$ in (5) by any real number $r \in (-1,1)$. The case $r = 0$ yields trivial results. One could use the linear recurrence relation $c(m+1,n+1) = \zeta c(m+1,n) + \bar{\zeta} c(m,n+1) - c(m,n)$ where ζ is a complex number and $|\zeta| < 1$. One could also use the recurrence relation $\mathcal{R}_{r,s}$ defined on $\mathbb{Z}^2$ by:

$$\begin{aligned} c(m+2,n+1) - rc(m+1,n+1) - sc(m,n+1) \\ = sc(m+2,n) + rc(m+1,n) - c(m,n) \end{aligned}$$

where r and s are two real numbers such that $|r| + |s| < 1$. One of the crystalline measures constructed by Olevskii and Ulanovskii in [16] can be obtained from a bounded solution of the linear recurrence relation defined on $\mathbb{Z}^2$ by:

$$c(m,n+1) - c(m,n-1) = rc(m+1,n) - rc(m-1,n) \tag{6}$$

where $(m,n) \in \mathbb{Z}^2$ and $r \in (-1,1)$.

The first part of this note is devoted to the study of (5). In a second part (6) is investigated and the results obtained by Olevskii and Ulanoskii are recovered. Then other constructions of crystalline measures are discussed.

Let κ be defined on $\mathbb{Z}^2$ by

$$\kappa(0,0) = 1,\ \kappa(1,0) = -1/2,\ \kappa(0,1) = -1/2,\ \kappa(1,1) = 1, \tag{7}$$

and $\kappa = 0$ elsewhere on $\mathbb{Z}^2$. If the sequence $c(m,n)$, $(m,n) \in \mathbb{Z}^2$, is denoted by γ an equivalent formulation of (5) is given by the convolution equation $\gamma * \kappa = 0$ on $\mathbb{Z}^2$. A similar remark is valid for the other examples of linear recurrence equation which are studied in this note. We have seen that the solutions of a linear recurrence relation on a lattice $\mathcal{L}$ are not almost-periodic sequences in general. But they are always the mean-periodic functions on $\mathcal{L}$. Here we anticipate on Section 6.

There exists a non trivial solution $c(m,n)$ to (5) such that $c(m,0) = 0$ identically in $m \in \mathbb{Z}$. To construct this solution it suffices to impose $c(m,0) = 0$ for $m \in \mathbb{Z}$, $c(0,1) = 1$, and $c(0,n) = 0$ for $n \in \mathbb{Z}$, $n \neq 1$. All other values of $c(m,n)$ are then deduced from (5). First we have $c(m,n) = 0$ for any m and any $n \leq 0$. From the point

$(1,0)$ we move up and right. We obtain $c(1,1) = 1/2, c(2,1) = 1/4, c(3,1) = 1/8, \ldots$. Moving up and left we obtain $c(-1,1) = 2, c(-2,1) = 4, c(-3,1) = 8, \ldots$. We then keep moving up and starting from the column $m = 0$ we compute $c(m,n)$ for any $n \geq 2$. This solution to (5) has an exponential growth. Another example of a solution of (5) with exponential growth is given by $c(m,n) = (7/2)^m(1/4)^n$. We give up these examples and study the solutions to (5) belonging to $l^\infty(\mathbb{Z}^2)$. Some bounded solutions of (5) are provided by the following lemma:

Lemma 1. *For any "initial condition" $a_0(m) \in l^2(\mathbb{Z})$ there exists a unique solution $c(m,n) \in l^\infty(\mathbb{Z}^2)$ to (5) such that $c(m,0) = a_0(m)$. Moreover this solution tends to 0 as $|m| + |n|$ tends to infinity.*

The following definition is needed in the proof of Lemma 1.

Definition 1. *One denotes by $\mathbf{T}$ the circle group and by $\mathcal{S}'(\mathbf{T})$ the space of all distributions on $\mathbf{T}$. Then $PM(\mathbf{T})$ is the space of all $\sigma \in \mathcal{S}'(\mathbf{T})$ whose Fourier coefficients belong to $l^\infty(\mathbb{Z})$. The norm $\|\sigma\|_{PM}$ is the l^∞ norm of these Fourier coefficients.*

Let us study the solutions $c(m,n) \in l^\infty(\mathbb{Z}^2)$ to (5). We consider the Fourier series

$$f_n(z) = \sum_{m=-\infty}^{\infty} c(m,n) z^m \tag{8}$$

where $z = \exp(i\theta)$, $\theta \in [0, 2\pi)$. Then f_n belongs to $PM(\mathbf{T})$ and (5) is equivalent to the recurrence relation:

$$(1 - z/2) f_{n+1}(z) = (1/2 - z) f_n(z),\ n \in \mathbb{Z}. \tag{9}$$

The inner function $B(z) = \frac{1/2 - z}{1 - z/2}$ is holomorphic in the unit disc, it satisfies $|B(z)| = 1$ and $B^{-1}(z) = B(\overline{z})$ on the circle group. We then have $f_{n+1}(z) = B(z) f_n(z)$ and $f_n(z) = B^n(z) f_0(z)$, $n \in \mathbb{Z}$. Let $L^2(\mathbf{T})$ be the Hilbert space of square integrable functions on $\mathbf{T}$ equipped with the norm $\|f\|_2 = (\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta/2\pi)^{1/2}$. Since f_0 belongs to $L^2(\mathbf{T})$ it implies that f_n also belongs to $L^2(\mathbf{T})$ and we have $\|f_n\|_2 = \|f_0\|_2$ for any $n \in \mathbb{Z}$. This ends the proof of the first assertion of Lemma 1. The same proof can be used for the recurrence relation $\mathcal{R}_{r,s}$. Then $B(z)$ is replaced by $C(z) = \frac{z^2 - rz - s}{-sz^2 - rz + 1}$. We have $|C(z)| = 1$ on the circle group.

Let us prove the second assertion of Lemma 1. It relies on the following lemma:

Lemma 2. *If $B(z) = \frac{1/2 - z}{1 - z/2}$ there exists a constant C such that $\|B^n\|_{PM(\mathbf{T})} \leq C|n|^{-1/3}$ for $n \in \mathbb{Z}$, $|n| \geq 1$.*

Indeed, $B(\exp(i\theta)) = \exp(i\xi(\theta))$ where ξ is the real valued $\mathcal{C}^\infty$ function defined by $\xi(0) = \pi$ and $\frac{d\xi}{d\theta} = \frac{3/4}{5/4 - \cos\theta}$. The Fourier coefficients of $B^n(z)$ are

$$I(m,n) = \frac{1}{2\pi} \int_0^{2\pi} \exp(in\xi(\theta) - im\theta)\, d\theta, \tag{10}$$

and we shall estimate $\sup_{m \in \mathbb{Z}} |I(m,n)|$. Then Lemma 2 follows from van der Corput's lemma as it is shown now. The function ξ is increasing and we have $\xi(\theta + 2\pi) = \xi(\theta) - 2\pi$. The second derivative ξ'' of ξ vanishes when $\theta = 0$ or $\theta = \pi$ but we have

$|\xi''(\theta)| \geq c_1(\eta) > 0$ if $\eta \leq |\theta| \leq \pi - \eta$ and $0 < \eta \leq \pi/6$. The standard stationary phase estimate cannot be used but the third derivative of ξ satisfies $|\xi'''(\theta)| \geq c_2(\eta) > 0$ if $|\theta| \leq \eta$ or $|\theta - \pi| \leq \eta$ and if $\eta > 0$ is small enough. Finally, van der Corput's lemma implies the uniform estimate $|I(m,n)| \leq C|n|^{-1/3}$. Let us observe that the standard stationary phase estimate $|I(m,n)| \geq c|n|^{-1/2}$ is valid if $|m| \leq |n|/4$ and implies that the norm of $B(z)^n$ in the Wiener algebra $A(\mathbf{T})$ is larger than $c'\sqrt{|n|}$. The Wiener algebra consists of the continuous functions f on $\mathbf{T}$ whose Fourier coefficients $\widehat{f}(k), k \in \mathbb{Z}$, belong to $l^1(\mathbb{Z})$ and the norm of f in the Wiener algebra is this l^1 norm [18].

To prove the second assertion of Lemma 1 we first treat the case where $a_0(0) = 1$ and $a_0(m) = 0$ if $m \neq 0$. Then Lemma 2 and (9) imply that the solution of (5) tends to 0 at infinity. The space of all solutions of (5) is translation invariant and the same conclusion holds if $a_0(m) = 0$ if $m \neq m_0$ and $a_0(m_0) = 1$. If a_0 is finitely supported the same conclusion is obtained by linearity. Finally the general case follows by density in $l^2(\mathbb{Z})$.

It is tempting to replace the hypothesis $a_0(m) \in l^2(\mathbb{Z})$ by the more natural condition $a_0(m) \in l^\infty(\mathbb{Z})$. Unfortunately, Lemma 1 would be wrong since the norm of $B(z)^n$ in the Wiener algebra $A(\mathbf{T})$ growths as $\sqrt{n}$ as $n \to \infty$. Let $B(\mathbb{Z})$ be the Banach algebra consisting of Fourier–Stieltjes coefficients $\widehat{\mu}(k), k \in \mathbb{Z}$, of complex valued Radon measures μ on the circle group $\mathbf{T}$. The norm of $\widehat{\mu}$ in $B(\mathbb{Z})$ is $\|\mu\|$. Then $B(\mathbb{Z})$ contains $l^2(\mathbb{Z})$ and is contained in $l^\infty(\mathbb{Z})$. Similarly, $B(\mathbb{Z}^2)$ denotes the Banach algebra of Fourier–Stieltjes coefficients of the complex Radon measures μ on the two dimensional torus $\mathbf{T}^2$. The norm of $\widehat{\mu}$ in $B(\mathbb{Z}^2)$ is the total mass of $|\mu|$ denoted by $\|\mu\|$. We then have:

Theorem 1. *For any initial condition $a_0(m) \in B(\mathbb{Z})$ there exists a unique solution $c(m,n) \in l^\infty(\mathbb{Z}^2)$ to (5) such that $c(m,0) = a_0(m)$. We have $c(m,n) \in B(\mathbb{Z}^2)$ and $\|c(m,n)\|_{B(\mathbb{Z}^2)} = \|a_0(m)\|_{B(\mathbb{Z})}$. Moreover if $a_0(m)$ are the Fourier–Stieltjes coefficients of a probability measure ρ on $\mathbf{T}$, then $c(m,n)$ are the Fourier–Stieltjes coefficients of a probability measure σ on the two dimensional torus $\mathbf{T}^2$.*

These probability measures σ play a seminal role in our second construction of sparse crystalline measures given in Section 5. Let us define $f_n(z)$ by (8). If $f_0(z) = \delta_{z_0}(z)$ is the Dirac measure at $z_0 \in \mathbf{T}$ then we obviously have $f_n(z) = B^n(z_0)\delta_{z_0}(z)$, $n \in \mathbb{Z}$, which implies $c(m,n) = \overline{z}_0{}^m B^n(z_0)$. Since $|B(z_0)| = 1$, σ is a Dirac measure at $(z_0, B(\overline{z}_0))$. This remark suggests that for any probability measure ρ on $\mathbf{T}$, σ is the image of ρ by the map $z \to (z, B(\overline{z}))$. This claim will be proved in Section 5. Let $z = \exp(i\theta)$ and $\psi(\theta) = -\xi(\theta)$. We then have $B(\overline{z}) = \exp(i\psi(\theta))$ and the point $(z, B(\overline{z}))$ belongs to the curve Γ pictured in Figure 1. This function ψ will be met again in Section 5.

Here are some preliminary remarks needed in our first construction of sparse crystalline measures. Let $S \subset \mathbb{Z}^2$ be the sector defined by $m \geq 0, n \geq 0$. It is easy to construct non trivial solutions $c \in l^\infty(\mathbb{Z}^2)$ of (5) which are supported by $S + B$ where B is a compact disc. Then it is proved in Section 4 that the series $\nu = \sum_{(m,n)\in\mathbb{Z}^2} c(m,n)\delta_{(m+n\alpha)}$ is a crystalline measure when $\alpha > 0$. Moreover, the support Λ of the inverse Fourier transform of ν only depends on α and not of the solution $c(m,n)$ of (5). This support Λ is a uniformly discrete set of real numbers.

Here is the construction of these localized solutions. If the initial condition $c(m,0) \in l^2(\mathbb{Z})$ satisfies $c(m,0) = 0$ for $m \le -1$ then f_0 belongs to the Hardy space $\mathcal{H}^2$. Therefore the same is true for f_n, $n \in \mathbb{N}$, since $B(z)$ is holomorphic in the unit disc. In other terms for any $n \in \mathbb{N}$ we have $c(m,n) = 0$ for $m \le -1$. Moreover $c(0,n) = 2^{-n}c(0,0)$. Similarly, $c(m,0) = 0$ for $m \ge 1$ implies $c(m,n) = 0$ for $n \le -1$ and $m \ge 1$. These two remarks and the translation invariance of (5) imply the following. If $f_0 = \sum_{-T}^{T} c(m,0)z^m$ is a trigonometric polynomial then $c(m,n) = 0$ if $n \ge 0, m < -T$ while $c(m,n) = 0$ if $m > T, n \le 0$. Here is a sharper result.

Lemma 3. *Let $S \subset \mathbb{Z}^2$ be the sector defined by $m \ge 0, n \ge 0$. Let $\mathcal{V}$ be the vector space of all bounded solutions to* (5) *which are supported by $S \cup (-S)$. Then the dimension of $\mathcal{V}$ is* 3.

Indeed, one is looking for an initial condition $f_0 \in PM(\mathbf{T})$ such that $a(z) = B(z)f_0(z)$ is holomorphic in the unit disc and $b(z) = B^{-1}(z)f_0(z)$ is anti-holomorphic. It implies $(1 - z/2)^2 a(z) = (1 - 2z)^2 b(z)$. Therefore the Fourier series of $E(z) = (1 - z/2)^2 a(z)$ is $\sum_0^\infty a_k z^k$ but is also $(1 - 2z)^2 b(z) = \sum_{-\infty}^{2} b_k z^k$. It implies $E(z) = a_0 + a_1 z + a_2 z^2$. Finally the dimension of $\mathcal{V}$ is at most 3. We now construct a basis of $\mathcal{V}$.

We first study (5) with the initial condition $c(0,0) = 1$ and $c(m,0) = 0$ if $m \ne 0$. Then $f_n(z) = B(z)^n$ for every $n \in \mathbb{Z}$. When $n \ge 0$, f_n is holomorphic and $c(m,n) = 0$ for $m \le -1$. When $n \le -1$, $f_n(z) = B(\overline{z})^{|n|}$ is anti-holomorphic and $c(m,n) = 0$ for $m \ge 1$. Moreover $c(0,n) = 2^{-|n|}$ and $c(-m,-n) = c(m,n)$ on $\mathbb{Z}^2$.

Definition 2. *This solution to* (5) *is denoted by $\gamma_0(m,n)$.*

In contrast with what happens in the one dimensional case γ_0 is a bounded solution to (5) which is not Bohr almost periodic. A simple calculation gives $\sum_m \gamma_0(n-m,m) = \cos(n\pi/3)$ for any $n \in \mathbb{Z}$.

In our second example $c(m,0) = (1/2)^m$ if $m \ge 0$ and $c(m,0) = 0$ if $m \le -1$. Then $f_0(z) = (1 - z/2)^{-1}$ and $f_n(z)$ is holomorphic if $n \in \mathbb{N}$. We have $c(0,n) = (1/2)^n$, $n \ge 0, c(0,n) = 0$, $n \le -1$. Moreover $f_{-1}(z) = -\frac{\overline{z}}{1-\overline{z}/2}$ and more generally $f_n(z) = -B^{|n|}(\overline{z})\frac{\overline{z}}{1-\overline{z}/2}$ if $n \le -1$. Therefore, f_n is anti-holomorphic if $n \le -1$. We have $c(-m,-n) = -c(m-1,n-1)$ on $\mathbb{Z}^2$ in this second example.

Definition 3. *This solution to* (5) *is denoted by $\gamma_1(m,n)$.*

We have $\sum_m \gamma_1(n-m,m) = \sin((n+1)\pi/3)/\sin(\pi/3))$ for any $n \in \mathbb{Z}$. In our third example $c(m,0) = (1/2)^{|m|}$ if $m \le 0$ and $c(m,0) = 0$ if $m \ge 1$. Then $f_0(z) = (1-\overline{z}/2)^{-1}$. We have $f_1(z) = B(z)f_0(z) = -z(1-z/2)^{-1}$ and $f_n(z)$ is holomorphic if $n \in \mathbb{N}$. We have $c(0,n) = 0$, $n \ge 1, c(0,n) = (1/2)^{|n|}$, $n \le 0$. Moreover $f_{-1}(z) = \frac{1/2-\overline{z}}{(1-\overline{z}/2)^2}$. Therefore f_n is anti-holomorphic if $n \le -1$.

Definition 4. *This solution to* (5) *is denoted by $\gamma_2(m,n)$.*

We have $\sum_m \gamma_2(n-m,m) = -\sin((n-1)\pi/3)/\sin(\pi/3))$ for any $n \in \mathbb{Z}$. These three solutions are linearly independent and the existence of sparse crystalline measures follows directly from the properties of these three solutions of (5). It can be proved that any solution of (5) which is supported by $S \cup (-S)$ is a bounded solution.

We now consider the linear recurrence relation (6). This example differs strongly from the preceding one. As it was mentioned above we start with a real number $r \in (-1, 1)$ and consider the linear recurrence relation defined on $\mathbb{Z}^2$ by:

$$c(m, n+2) - c(m, n) = r[c(m+1, n+1) - c(m-1, n+1)]. \tag{12}$$

The analysis we performed on (5) is used here. Instead of the complex variable z used in (8) it is more convenient to set $z = \exp(ix)$ and to use the real variable x. We consider the 2π-periodic functions f_n, $n \in \mathbb{Z}$, defined by

$$f_n(x) = \sum_m c(m, n) \exp(imx). \tag{13}$$

Then (12) is equivalent to $f_{n+2}(x) + 2ir \sin x\, f_{n+1}(x) - f_n(x) = 0$. Let us define the function $\phi(x)$ by $\phi(x) \in [-\pi/2, \pi/2]$ and $\sin\phi = r \sin x$. We then have

$$f_n(x) = a(x) \exp(-in\phi) + (-1)^n b(x) \exp(in\phi),\ n \in \mathbb{Z}. \tag{14}$$

Lemma 4. *If the initial conditions $a_0(m)$ and $a_1(m)$ both belong to $l^2(\mathbb{Z})$ there exists a unique bounded solution to* (12) *such that $c(m, 0) = a_0(m)$ and $c(m, 1) = a_1(m)$, $m \in \mathbb{Z}$.*

Let us find $a(x)$ and $b(x)$ such that

$$f_0(x) = a(x) + b(x),\ \ f_1(x) = a(x) \exp(-i\phi(x)) - b(x) \exp(i\phi(x)).$$

It yields $f_0(x) \exp(i\phi(x)) + f_1(x) = 2a(x) \cos(\phi(x))$. But $\cos(\phi(x)) \geq \sqrt{1 - r^2}$ implies $a(x) \in L^2(0, 2\pi)$. Similarly, $b(x) \in L^2(0, 2\pi)$. Finally, (14) implies $\|f_n\|_2 \leq C$ as announced.

We consider the solution to (12) defined by $f_0(x) = 1$ and $f_1(x) = 0$. We then have $f_n(x) = \frac{\cos(n-1)\phi}{\cos\phi}$ if n is even and $f_n(x) = -i\frac{\sin(n-1)\phi}{\cos\phi}$ if n is odd. It implies $f_{n+2} + f_n(x) = 2\cos(n\phi)$ if n is even and $f_{n+2} + f_n(x) = -2i \sin n\phi$ if n is odd. It follows from the properties of Chebyshev polynomials that $f_n(x)$ is a trigonometric polynomial whose highest frequency is $n - 2$ if $n \geq 2$ and $|n|$ if $n \leq -1$. It implies that the support of $c(m, n)$ is contained in the cone $|n| \geq |m|$.

Definition 5. *This solution to* (12) *is denoted by $\alpha(m, n)$.*

It is proved in Section 4 that $\nu = \sum_{(m,n)\in\mathbb{Z}^2} \alpha(m, n)\delta_{(m+n\alpha)}$ is a crystalline measure when $\alpha > 1, \alpha \notin \mathbb{Q}$, and $\alpha > \frac{r}{\sqrt{1-r^2}}$.

3. Crystalline measures

Some usual notations and definitions are recalled in this introductory section. The Dirac measure at $a \in \mathbb{R}^n$ is denoted by δ_a or $\delta_a(x)$. A *purely atomic measure* is a linear combination $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ of Dirac measures where the coefficients $c(\lambda)$ are real or complex numbers and $\sum_{|\lambda|\leq R} |c(\lambda)|$ is finite for every $R > 0$. If $c(\lambda) \neq 0$, $\forall\lambda \in \Lambda$, then Λ is the *support* of μ. A subset $\Lambda \subset \mathbb{R}^n$ is *locally finite* if $\Lambda \cap B$ is finite for every bounded set B. Equivalently, Λ can be ordered as a sequence $\{\lambda_j,\ j = 1, 2, \dots\}$ such that $|\lambda_j|$ tends to infinity with j. A measure μ is a *tempered distribution* if it has a polynomial growth at infinity in the sense given by Laurent Schwartz in [19]. For instance, the measure $\sum_{k=1}^{\infty} 2^k \delta_k$ is not a tempered distribution, while the two series $\sum_{k=1}^{\infty} k^3 \delta_k$ and $\sum_{k=1}^{\infty} 2^k [\delta_{(k+2^{-k})} - \delta_k]$ are tempered distributions. These two examples illustrate the following proposition.

Proposition 1. *Let $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda$ be an atomic measure supported by a uniformly discrete set $\Lambda \subset \mathbb{R}$. Then μ is a tempered distribution if and only if there exist an exponent N and a constant C such that $|a(\lambda)| \le C(1 + |\lambda|)^N$, $\lambda \in \Lambda$.*

The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-ix \cdot y) f(x)\, dx$. The distributional Fourier transform $\widehat{\mu}$ of μ is defined by the condition that $\langle \widehat{\mu}, \phi \rangle = \langle \mu, \widehat{\phi} \rangle$ holds for every test function ϕ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Definition 6. *A purely atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if:*

(a) *the support Λ of μ is a locally finite set;*
(b) *μ is a tempered distribution;*
(c) *the distributional Fourier transform $\widehat{\mu}$ of μ is also a purely atomic measure supported by a locally finite set S.*

A crystalline measure μ is sparse if the support Λ of μ is uniformly discrete.

A set Λ of real numbers is uniformly discrete if there exists a $\beta > 0$ such that $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$ imply $|\lambda' - \lambda| \ge \beta$. In other terms, Λ can be ordered as an increasing sequence λ_j, $j \in \mathbb{Z}$, of real numbers such that $\lambda_{j+1} - \lambda_j \ge \beta$ for any $j \in \mathbb{Z}$. If moreover there exists a constant C such that $\lambda_{j+1} - \lambda_j \le C$ for any $j \in \mathbb{Z}$, then Λ is a Delone set.

Let μ be a crystalline measure. We then have $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda$ and $\nu = \widehat{\mu} = \sum_{y \in S} b(y)\delta_y$ where Λ, S are two locally finite sets. Then for every test function $f \in \mathcal{S}(\mathbb{R}^n)$ the following *generalized Poisson formula* holds

$$\sum_{\lambda \in \Lambda} a(\lambda)\widehat{f}(\lambda) = \sum_{y \in S} b(y) f(y). \tag{15}$$

This can also be written

$$\sum_{\lambda \in \Lambda} a(\lambda) \exp(ix \cdot \lambda) = \sum_{y \in S} b(y)\delta_y. \tag{16}$$

It leads to the following conjecture:

Conjecture 1. *The series $\sum_{\lambda \in \Lambda} a(\lambda) \exp(ix \cdot \lambda)$ is the Fourier series of the almost periodic distribution $\nu = \sum_{y \in S} b(y)\delta_y$.*

A tempered distribution S is almost periodic if for any test function ϕ the convolution product $T * \phi$ is a Bohr almost periodic function [19]. An almost periodic measure ν is defined by a similar condition where ϕ is any compactly supported continuous function. Proposition 1 implies Conjecture 2 if Λ is uniformly discrete. Indeed, the inverse Fourier transform of $\nu * \phi$ is the atomic measure $\rho = g\mu$ where g is the inverse Fourier transform of ϕ. Proposition 1 implies that the total mass of ρ is finite. Therefore $\nu * \phi$ is an almost periodic function with an absolutely convergent Fourier series. In general, ν is not an almost periodic measure. It is not even translation bounded, as it is shown in Section 8. But ν is a mean-periodic measure if Λ is uniformly discrete (Theorem 6). Theorem 6 opens the gate to the best available tool for constructing sparse crystalline measures (Theorem 5).

If μ is a crystalline measure, so is its distributional Fourier transform. This symmetry is not satisfied in general for sparse crystalline measures. The simplest example of a crystalline measure is the Dirac comb $\mu = \sum_{k=-\infty}^{\infty} \delta_k$ whose support is $\mathbb{Z}$. The measure $\mu_{(a,b)} = \sum_{k=-\infty}^{\infty} \delta_{ak+b}$, $a > 0, b \in \mathbb{R}$, is a Dirac comb supported by $a\mathbb{Z} + b$. If μ is the Dirac comb its Fourier transform is a Dirac comb supported by $2\pi\mathbb{Z}$. Simple manipulations on Dirac combs yield other examples of crystalline measures.

Definition 7. *Let σ_j, $1 \le j \le N$, be a Dirac comb supported by $\Gamma_j = a_j\mathbb{Z} + b_j$ and let $g_j(x)$, $1 \le j \le N$, be a finite trigonometric sum. Let μ_j be the product $g_j\,\sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ is called a generalized Dirac comb.*

The support of a generalized Dirac comb μ is a locally finite set since it is included in $\cup_1^N \Gamma_j$. The Fourier transform of a generalized Dirac comb is a generalized Dirac comb. Therefore a generalized Dirac comb is a crystalline measure. In one dimension Lev and Olevskii proved in [7] that if μ and its distributional Fourier transform are sparse crystalline measures, then μ is a generalized Dirac comb. Do other crystalline measures exist? André-Paul Guinand (1912–1987) pioneered this investigation in [1] and proposed several examples of non trivial crystalline measures. Here is one of his examples. One defines $\chi \colon \mathbb{Z}^3 \to \{-1/2, 0, 4\}$ by $\chi(k) = 0$ if $k \in 4\mathbb{Z}^3$, $\chi(k) = 4$ if $k \in 2\mathbb{Z}^3 \setminus 4\mathbb{Z}^3$ and $\chi(k) = -1/2$ if $k \in \mathbb{Z}^3 \setminus 2\mathbb{Z}^3$. Then the Fourier transform of the one-dimensional odd measure $\tau = \sum_{k\in\mathbb{Z}^3} \chi(k)|k|^{-1}(\delta_{|k|/2} - \delta_{-|k|/2})$ is $-i\tau$. The support of Guinand's crystalline measure τ is contained in the set $\Lambda = \{\pm\sqrt{n};\, n \in \mathbb{N}\}$. The proofs of Guinand's claims can be found in [14] and [15].

4. A sparse crystalline measure

If μ is a crystalline measure, its distributional Fourier transform $\widehat{\mu}$ is also a crystalline measure. This symmetry between μ and its Fourier transform $\widehat{\mu}$ shall be broken if we want to construct a non trivial sparse crystalline measure. Indeed, we know from [7] that a sparse crystalline measure μ whose distributional Fourier transform is also sparse is a generalized Dirac comb. These remarks force us to make a choice in our construction. Either we focus on μ which is the case in our second and fourth proof. The support Λ of μ is then a curved model set. Or we focus on $\widehat{\mu}$ which is the choice made in our first and third proof. Then the theory of mean-periodic functions is our main tool.

Our first construction of a sparse crystalline measure relies on the properties of the atomic measure on the real line defined by

$$\nu_0 = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_0(m,n)\delta_{m+n\sqrt{2}}. \tag{17}$$

In our construction $\sqrt{2}$ can be replaced by any $\alpha > 0, \alpha \notin \mathbb{Q}$. The corresponding series where $\gamma_0(m,n)$ is replaced by $\gamma_1(m,n)$ and $\gamma_2(m,n)$ are also used. The corresponding measures are denoted by ν_1 and ν_2.

The right-hand side of (17) makes sense. Indeed, we know that $\gamma_0(m,n) = 0$ unless m and n have the same sign. Then $|m + n\sqrt{2}| = |m| + |n\sqrt{2}|$. Therefore, the series in (17) is locally finite. Since γ_0 belongs to $l^\infty(\mathbb{Z}^2)$ the measure ν_0 is a tempered distribution. Indeed, $|\nu_0|([-R,R]) \leq C_0 R^2$. We have

$$\nu_0 = \sum_{m\geq 0} \sum_{n \geq 0} \gamma_0(m,n)(\delta_{m+n\sqrt{2}} + \delta_{-m-n\sqrt{2}}) - \delta_0 \tag{18}$$

and ν_0 is an even measure.

Theorem 2. *The atomic measure ν_0 is a crystalline measure and the support of the distributional Fourier transform μ_0 of ν_0 is a uniformly discrete set.*

The distributional Fourier transform of ν_0 exists since ν_0 is a tempered distribution. The following lemma is the core of the proof of Theorem 2.

Lemma 5. *Let $\tau = \delta_0 - (1/2)\delta_1 - (1/2)\delta_{\sqrt{2}} + \delta_{1+\sqrt{2}}$. We then have*

$$\nu_0 * \tau = 0. \tag{19}$$

Indeed, γ_0 satisfies $\gamma_0 * \kappa = 0$ which is identical to (19). The definition of κ is given in (7).

We now prove Theorem 2. The Fourier transform of a convolution product is an ordinary product and Lemma 5 implies $\widehat{\tau}\,\mu_0 = 0$. Therefore it suffices to check that the real zeros of $\widehat{\tau}$ are simple to prove that the tempered distribution μ_0 is an atomic measure. It suffices to check that the set Λ of these real zeros is uniformly discrete to prove that μ_0 is a sparse crystalline measure. It is proved in Section 7 that the zeros of the Fourier-Laplace transform of τ are real. This information is not needed here. The following lemmas take care of these two verifications.

Lemma 6. *We have $|\frac{d}{dt}\widehat{\tau}(t)| \geq \frac{1+\sqrt{2}}{2}$ uniformly on the real line.*

Let $z = \exp(-it)$ and $w = \exp(-i\sqrt{2}t)$. Then $|\frac{d}{dt}\widehat{\tau}(t)| = |(1/2)z + (\sqrt{2}/2)w - (1 + \sqrt{2})zw| \geq \frac{1+\sqrt{2}}{2}$ since $|w| = |z| = 1$. Lemma 6 implies that the zeros of $\widehat{\tau}$ are simple. We now use a well known observation.

Lemma 7. *If a complex valued function g belongs to $\mathcal{C}^2(\mathbb{R})$ and satisfies $|\frac{d}{dt}g(t)| \geq 1$ and $|\frac{d^2}{dt^2}g(t)| \leq C$ then the set of zeros of g is uniformly discrete.*

It suffices to prove that $g(t_0) = 0$ implies $g \neq 0$ on $(t_0, t_0 + C^{-1}]$. To check this claim we write $g(t) = g(t_0) + (t - t_0)g'(t_0) + \int_{t_0}^t (t-s)g''(s)ds$. It implies $|g(t)| \geq t - t_0 - C(t-t_0)^2/2 \geq (t-t_0)/2$. This ends the proof of Lemma 7 and of Theorem 2.

Property (19) implies that ν_0 is a mean-periodic measure. This is not accidental. The distributional Fourier transform of a crystalline measure supported by a uniformly discrete set is a mean-periodic measure. This will be proved in Section 8 and explains our strategy. Olevskii and Ulanovskii recently proved the following result [17]: If $\Lambda \subset \mathbb{R}$ is uniformly discrete, if $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a crystalline measure and if $|\widehat{\sigma}_\Lambda|$ has a polynomial growth at infinity, then there exists a finitely supported measure τ such that $\tau * \widehat{\sigma}_\Lambda = 0$.

The argument used in the proof of Theorem 2 remains valid if $\gamma_0(m,n)$ is replaced by the sequence α defined by (12). If $\alpha > 1, \alpha \notin \mathbb{Q}$, and $\alpha > \frac{r}{\sqrt{1-r^2}}$ the atomic measure $\nu = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \alpha(m,n)\delta_{m+n\alpha}$ is a crystalline measure and its distributional Fourier transform is supported by a uniformly discrete set as it is stated in [16].

5. A second proof

In the proof of Theorem 2 given in Section 4 the story begins in the spectral domain. We started with the distributional Fourier transform ν_0 of the sparse crystalline measure μ_0. Then μ_0 is recovered by an inverse Fourier transform. This does not provide us with a direct access to the geometrical structure of μ_0. We only know that μ_0 is a linear combination of Dirac measures but we do not have any information on the coefficients. In the second proof of Theorem 2 we start from a geometrical definition of μ_0 and prove that its distributional Fourier transform ν_0 is an atomic measure supported by a locally finite set. This second proof was already detailed in 7cite12. The support Λ of μ_0 is a curved model set and the geometrical structure of Λ is obvious from this definition.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The two coordinates on $\mathbb{T}^2$ will be denoted by θ and ϕ. Let $\Gamma \subset \mathbb{T}^2$ be the curve whose equation is $1 - (1/2)\exp(i\theta) - (1/2)\exp(i\phi) + \exp(i(\theta+\phi)) = 0$. This equation can be written $\tan(\phi/2)\tan(\theta/2) = 1/3$. The curve Γ (see Figure 1) is the graph of the function $\psi(\theta) = 2\arctan[(1/3)\cot(\theta/2)]$. This only defines $\psi(\theta)$ modulo 2π. A better definition of the real valued function ψ is given by $\psi(0) = \pi$ and $\frac{d\psi}{d\theta} = \frac{3/4}{\cos\theta - 5/4}$. Therefore ψ is a decreasing function of θ. We have $\psi(\theta+2\pi) = \psi(\theta) - 2\pi$. This function ψ is the opposite of the one which was used in the proof of Lemma 2, Section 2. Moreover Γ is symmetric with respect to $(0,0)$ and with respect to the diagonal $\phi = \theta$ of $\mathbb{T}^2$. For any sequence $c(m,n) \in l^\infty(\mathbb{Z}^2)$ the double Fourier series

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c(m,n)\exp(im\theta + in\phi) \tag{22}$$

converges in the distributional sense to a distribution σ on $\mathbb{T}^2$. We have

Lemma 8. *The distribution*

$$\sigma_0 = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_0(m,n)\exp(im\theta + in\phi) \tag{23}$$

is a measure supported by Γ. The same result is true for σ_1 and σ_2.

Lemma 8 is fully consistent with Theorem 1. We now prove Lemma 8. As it was already observed (5) can be written as $\gamma_0 * \kappa = 0$ where $\kappa(0,0) = \kappa(1,1) = 1$ and $\kappa(1,0) = \kappa(0,1) = -1/2$ and $\kappa = 0$ elsewhere on $\mathbb{Z}^2$. Let $K(\theta,\phi) = \sum_{m,n} \kappa(m,n)\exp(im\theta + in\phi) = 1 - (1/2)\exp(i\theta) - (1/2)\exp(i\phi) + \exp(i(\theta+\phi))$. Moving to the Fourier domain $\gamma_0 * \kappa = 0$ implies that the product between the function K and the distribution σ_0 is identically 0. Therefore σ_0 is a simple layer distribution supported by the curve Γ. The following lemmas are used to compute the density of σ_0 with respect to the arc length on Γ.

Lemma 9. *Let S be a distribution S on $\mathbb{T}^2$ such that $KS = 0$. Then for any test function u on $\mathbb{T}^2$ we have $\langle S, u\rangle = \langle S, v\rangle$ where $v(\theta, \phi) = u(\theta, \psi(\theta))$.*

Indeed, the function $g = u - v$ vanishes on Γ by construction. Therefore $g = wK$ where w is a test function on $\mathbb{T}^2$. It implies $\langle S, u - v\rangle = \langle S, wK\rangle = \langle KS, w\rangle = 0$.

The image by the map $(\theta, \phi) \mapsto (\theta, 0)$ of a distribution S on $\mathbb{T}^2$ is a distribution T on $\mathbb{T}$. This distribution T is defined by $\langle T, u\rangle = \langle S, u\rangle$ where $u = u(\theta)$ is a test function depending only on the variable θ. If $\sum_{k,l} c(k, l) \exp(i(k\theta + l\phi))$ is the Fourier series of S then the Fourier series of T is $\sum_k c(k, 0) \exp(ik\theta)$.

Lemma 10. *Let S be a distribution S on $\mathbb{T}^2$ such that $KS = 0$. Let T be the image of S by the map $(\theta, \phi) \mapsto (\theta, 0)$. Then the distribution S is the image of the distribution T by the map $\theta \mapsto (\theta, \psi(\theta))$.*

Using the notations of Lemma 9 we have $\langle S, u\rangle = \langle S, v\rangle = \langle T, v\rangle$ which ends the proof.

Returning to the computation of σ_0 we have $T = \sum_k c(k, 0) \exp(ik\theta) = 1$ identically. Therefore, σ_0 is the image of the measure $2\pi d\theta$ on $[0, 2\pi]$ by the map $\theta \mapsto (\theta, \psi(\theta))$. The measure $\frac{1}{4\pi^2}\sigma_0$ is a probability measure in agreement with Theorem 1. For any continuous function u on $\mathbb{T}^2$ we have $\langle \sigma_0, u\rangle = 2\pi \int_{\mathbb{T}} u(\theta, \psi(\theta))d\theta$. The discussion of the second example is similar, and, using again the notations of (8), we end with $f_0(\theta) = (1- (1/2) \exp(i\theta))^{-1}$. Therefore, σ_1 is the image of the measure $2\pi(1 - (1/2) \exp(i\theta))^{-1}d\theta$ on $[0, 2\pi]$ by the map $\theta \mapsto (\theta, \psi(\theta))$. In the third example σ_2 is the image of the measure $2\pi(1 - (1/2) \exp(-i\theta))^{-1}d\theta$ on $[0, 2\pi]$ by the map $\theta \mapsto (\theta, \psi(\theta))$.

Let $h\colon \mathbb{R} \mapsto \mathbb{T}^2$ be the embedding of $\mathbb{R}$ into $\mathbb{T}^2$ defined by $h(t) = (t, \sqrt{2}t)$. We now reach the core of this second proof. We define μ_0 by

$$\mu_0 = \sigma_0 \circ h. \tag{24}$$

Let us sketch our second proof. The meaning and the computation of the right-hand side of (24) are given by some elementary differential geometry as it is shown below. It is easily proved that μ_0 is a linear combination of Dirac measures on a uniformly discrete set. Once this is achieved the Fourier series expansion of $\sigma_0 \circ h$ is deduced from the expansion of σ_0. Indeed we have $\sigma_0 = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_0(m, n) \exp(im\theta + in\phi)$ and formally it implies $\sigma_0 \circ h = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_0(m, n) \exp(imt + in\sqrt{2}t)$. Then the distributional Fourier transform of μ_0 is $2\pi \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_0(m, n)\delta_{m+n\sqrt{2}}$. Therefore μ_0 is a sparse crystalline measure.

Here are the details of this proof. Is it possible to define $\nu \circ h$ for any Radon measure ν on $\mathbb{T}^2$? This problem is illustrated by two simple examples. In the first example, ν is a continuous function F on $\mathbb{T}^2$. Then $F \circ h$ is obviously an almost periodic function on $\mathbb{R}$. If ν is an arbitrary Radon measure on $\mathbb{T}^2$ then $\nu \circ h$ does not always make sense. An example of this drawback is given by $\nu = \delta_0$. This being said, let us return to (24). The measure σ_0 is approximated by a smooth function σ_0^ϵ. Then μ_0^ϵ is defined by $\mu_0^\epsilon = \sigma_0^\epsilon \circ h$ and it suffices to pass to the limit as ϵ tends to 0 to give a meaning to (24).

The vector $\mathbf{e} = (1, \sqrt{2})$ is uniformly transverse to the curve Γ since the function $\psi(\theta)$ is decreasing. If n_x is the unit normal vector to Γ at $x \in \Gamma$ we have

$$|\mathbf{e} \cdot n_x| \geq 1 \tag{25}$$

on Γ. A narrow strip Γ_ϵ around Γ is defined by

$$\Gamma_\epsilon = \{x + (t, t\sqrt{2}); x \in \Gamma, |t| \leq \epsilon\}. \tag{26}$$

Then if ϵ is small enough the map $G\colon \mathbb{T} \times [-\epsilon, \epsilon] \mapsto \Gamma_\epsilon$ defined by $G(\theta, t) = (\theta + t, \psi(\theta) + t\sqrt{2})$ is a diffeomorphism. This is due to the transversality and the implicit function theorem. The Jacobian determinant of G is $|\psi'(\theta) - \sqrt{2}| \geq \sqrt{2}$. We denote by w an even non negative $\mathcal{C}^\infty$ function supported by the interval $[-1, 1]$ and such that $\int w = 1$. We set $w_\epsilon(t) = (1/\epsilon)w(t/\epsilon)$. The support of w_ϵ is the interval $[-\epsilon, \epsilon]$. Let τ_ϵ be the image of the probability measure $w_\epsilon(t)\, dt$ by $h\colon \mathbb{R} \mapsto \mathbb{T}^2$. Then the convolution product $\sigma_0^\epsilon = \sigma_0 * \tau_\epsilon$ is a $\mathcal{C}^\infty$ function on $\mathbb{T}^2$ and σ_0^ϵ is supported by Γ_ϵ. If $x = y + (s, s\sqrt{2})$ belongs to Γ_ϵ where $y = (\theta, \psi(\theta)) \in \Gamma$, $\theta \in \mathbb{T}$, and $|s| \leq \epsilon$, we have

$$\sigma_0^\epsilon(x) = \frac{w_\epsilon(s)}{|\psi'(\theta) - \sqrt{2}|}. \tag{27}$$

Let us define $\Lambda \subset \mathbb{R}$ by

$$\Lambda = \{t \in \mathbb{R}; h(t) \in \Gamma\}. \tag{28}$$

Since $G\colon \mathbb{T} \times [-\epsilon, \epsilon] \mapsto \Gamma_\epsilon$ is a diffeomorphism Λ is a Delone set. Moreover we have

$$\sigma_0^\epsilon \circ h = \sum_{\lambda \in \Lambda} \frac{w_\epsilon(t - \lambda)}{|\psi'(t) - \sqrt{2}|}. \tag{29}$$

We define $\sigma_0 \circ h$ as the limit of $\sigma_0^\epsilon \circ h$ when ϵ tends to 0. We obtain

$$\sigma_0 \circ h = \sum_{\lambda \in \Lambda} \omega_0(\lambda)\delta_\lambda \tag{30}$$

where

$$\omega_0(\lambda) = |\psi'(\lambda) - \sqrt{2}|^{-1}. \tag{31}$$

Let us observe that $\psi'(\lambda)$ is well defined since ψ' is a 2π-periodic function.

On the other hand the Fourier series of the $\mathcal{C}^\infty$ function σ_0^ϵ is

$$\sigma_0^\epsilon = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma_0(m, n)\widehat{w}(\epsilon(m + n\sqrt{2})) \exp(im\theta + in\phi). \tag{32}$$

This series is absolutely convergent. Therefore the expansion of $\sigma_0^\epsilon \circ h$ is

$$\sigma_0^\epsilon \circ h = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma_0(m, n)\widehat{w}(\epsilon(m + n\sqrt{2})) \exp(i(m + n\sqrt{2})t). \tag{33}$$

Since $|m + n\sqrt{2}| = |m| + |n|\sqrt{2}$ if $\gamma_0(m, n) \neq 0$ the series

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma_0(m, n) \exp(i(m + n\sqrt{2})t)$$

converges in the distributional sense. Both sides of (33) converge in the distributional sense when ϵ tends to 0. Passing to the limit we obtain

$$\sigma_0 \circ h = \sum_{\lambda \in \Lambda} \omega_0(\lambda)\delta_\lambda = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma_0(m,n) \exp(i(m + n\sqrt{2})t). \tag{34}$$

It ends the second proof of Theorem 2.

We have

$$\omega_0(\lambda) = \frac{1}{\sqrt{2} - \psi'(\lambda)} = \frac{5/4 - \cos\lambda}{\beta - 2\cos\lambda} \tag{35}$$

where $\beta = \sqrt{3}/4 + (5/4)\sqrt{2}$. If instead the sequence γ_1 is being used these coefficients are

$$\omega_1(\lambda) = \frac{\omega_0(\lambda)}{1 - (1/2)\exp(i\lambda)}, \tag{36}$$

and if we turn to γ_2 we obtain

$$\omega_2(\lambda) = \overline{\omega_1(\lambda)} = \frac{\omega_0(\lambda)}{1 - (1/2)\exp(-i\lambda)}. \tag{37}$$

But $5/4 - \cos\lambda = |1 - (1/2)\exp(i\lambda)|^2 = (1 - (1/2)\exp(i\lambda))(1 - (1/2)\exp(-i\lambda))$ which implies $\omega_1(\lambda) + \omega_2(\lambda) = \frac{2 - \cos\lambda}{\beta - 2\cos\lambda}$. If a and b satisfy $(5/4)a + 2b = \beta$ and $a + b = 2$ we have $a\omega_0(\lambda) + b(\omega_1(\lambda) + \omega_2(\lambda)) = 1$. We have proved the following:

Theorem 3. *If Λ is defined by* (28) *then $\sum_{\lambda \in \Lambda} \delta_\lambda$ is a crystalline measure.*

We now prove Theorem 1. Here are some preliminary remarks. Let ρ be a Radon measure on $\mathbb{T}$ and let R be the image of the measure ρ by the map: $\theta \mapsto (\theta, \psi(\theta))$. We have, for any continuous function u on $\mathbb{T}^2$,

$$\langle R, u \rangle = \int_{\mathbb{T}} u(\theta, \psi(\theta)) d\rho(\theta). \tag{38}$$

This measure R is supported by Γ which implies $KR = 0$. Therefore (5) is satisfied by the Fourier–Stieltjes coefficients $c(m,n) = \widehat{R}(m,n)$ of R. If u does not depend on the second variable ϕ we have

$$\langle R, u \rangle = \int_{\mathbb{T}} u(\theta) d\rho(\theta). \tag{39}$$

We now return to Theorem 1. The sequence $a_0(m)$, $m \in \mathbb{Z}$, is assumed to be the sequence of Fourier–Stieltjes coefficients of a Radon measure ρ. We then define R by (38). On the one hand $c(m,n) = \widehat{R}(m,n)$ satisfies (5). On the other hand (39) implies $c(m,0) = \widehat{R}(m,0) = \widehat{\rho}(m) = a_0(m)$. It ends the proof.

The treatment of (12) is similar. Indeed the curve defined by $\sin\phi = r\sin\theta$ has two components Γ_1 and Γ_2 (see Figure 2). The first component is defined by $\phi \in (-\pi/2, \pi/2)$ and the second one by $\phi \in (\pi/2, 3\pi/2)$. Then Γ_1 is the graph of ψ_1 and Γ_2 is the graph of ψ_2. The functions f_0 and f_1 defined by (13) are two Radon measures on $\mathbb{T}$. Let a and b be the two other Radon measures on $\mathbb{T}$ which are defined by (14). It then suffices to let R_1 be the image of the measure $a(\theta)$ by the map $\theta \mapsto (\theta, \psi_1(\theta))$ and to let R_2 be defined similarly by b and ψ_2. We set $R = R_1 + R_2$. Finally $c(m,n) = \widehat{R}(m,n)$ solves our problem.

6. Mean-periodic functions

A third proof of Theorem 2 is given in this section. This proof relies on the theory of mean-periodic functions. A Radon measure μ on $\mathbb{R}$ is almost periodic if for any compactly supported continuous function g the convolution product $\mu * g$ is a Bohr almost periodic function. Then for any Bohr almost periodic function f the product $f\mu$ is also an almost periodic measure. The Fourier coefficients $\widehat{\mu}(\omega)$ of an almost periodic measure are defined by

$$\widehat{\mu}(\omega) = \frac{\widehat{\mu * g}(\omega)}{\widehat{g}(\omega)} \tag{40}$$

for any compactly supported continuous function g such that $\widehat{g}(\omega) \neq 0$. The right-hand side of (40) does not depend on g. The spectrum S of μ is defined by $\widehat{\mu}(\omega) \neq 0$ and S is a numerable set. The proof of Theorem 2 we just gave exemplifies the following general fact.

Lemma 11. *Let μ be an almost periodic measure whose spectrum is contained in $\mathbb{Z} + \sqrt{2}\mathbb{Z}$. Then there exists a unique Radon measure σ on $\mathbb{T}^2$ such that $\widehat{\sigma}(p,q) = \widehat{\mu}(p + q\sqrt{2})$, $(p,q) \in \mathbb{Z}^2$.*

That is how the almost periodic measure μ_0 is related to σ_0 in Section 5. Is $\widehat{\mu_0}$ an almost periodic measure? The answer is no since $\widehat{\mu_0}$ is not even translation bounded as it is shown in Lemma 16. But $\widehat{\mu_0}$ is a mean-periodic measure and this fact paves the way to a deeper understanding of crystalline measures. Since $\widehat{\mu_0}$ is not an almost periodic measure, μ_0 cannot be a generalized Dirac comb.

We now forget these remarks and open the gate to a completely distinct proof. Let τ be a compactly supported Radon measure on the line. The measure which is identically 0 is excluded. We consider the convolution equation $f * \tau = 0$ where f is a continuous function of the real variable x. No growth condition at infinity is imposed on f. If f satisfies $f * \tau = 0$ one writes $f \in \mathcal{C}_\tau$. This vector space $\mathcal{C}_\tau$ is a Fréchet space when it is equipped with the topology of uniform convergence on compact intervals. The Fourier Laplace transform of τ is the entire function of the complex variable z defined by

$$\widehat{\tau}(z) = \int_{-\infty}^{\infty} \exp(-izx)\, d\tau(x). \tag{41}$$

This Fourier–Laplace transform generalizes the characteristic polynomial of a linear recurrence relation.

Definition 8. *Let τ be a compactly supported Radon measure and let $\Lambda \subset \mathbb{C}$ be defined by*

$$\Lambda = \{z \in \mathbb{C};\ \widehat{\tau}(z) = 0\}. \tag{42}$$

The measure τ is mild if (a) Λ is a uniformly discrete set of real numbers and (b) all the zeros of (42) are simple.

Here is an example.

Lemma 12. *Let τ be a real valued even measure supported by the interval $[-1,1]$. Let us assume that*

$$|\tau(\{1\})| > s = \int_{[0,1)} d|\tau|(t). \tag{43}$$

Then τ is a mild measure.

Two proofs of Lemma 12 are given. Without loosing generality it can be assumed that $\tau(\{1\}) = 1$. We have $\widehat{\tau}(z) = 2\cos z + 2\int_{[0,1)} \cos(zt)d\tau(t)$. We argue by contradiction and assume that $\widehat{\tau}(x_0 + iy_0) = 0$ for some $y_0 > 0$. The curve defined by $x \mapsto \cos(x + iy_0)$ is an ellipse E whose focal points are 1 and -1. The width of E is $2\cosh y_0$. The complex number $\cos(x_0 + iy_0)$ belongs to E. The complex number $\int_{[0,1)} \cos(z_0 t)d\tau(t)$ belongs to the convex domain D delimited by the homothetic ellipse sE. Since $0 < s < 1$ we have $D \cap E = \emptyset$. Therefore we cannot have $\widehat{\tau}(z_0) = 0$ and $y_0 > 0$. The same argument can be used if $y_0 < 0$. Here is a second proof. We argue by contradiction and assume that $\widehat{\tau}(x + iy) = 0$ for some real x and some $y > 0$. The case $y < 0$ would be identical. It implies

$$\cos x \cosh y + \int_{[0,1)} \cos(xt)\cosh(yt)d\tau(t) = 0, \tag{44}$$

and

$$\sin x \sinh y + \int_{[0,1)} \sin(xt)\sinh(yt)d\tau(t) = 0. \tag{45}$$

We combine linearly (44) and (45) with the coefficients $\cos x/\cosh y$ and $\sin x/\sinh y$. We obtain

$$1 + \int_{[0,1)} g(t)d\tau(t) = 0 \tag{46}$$

where

$$g(t) = \frac{\cos x \cos(xt)\cosh(yt)}{\cosh y} + \frac{\sin x \sin(xt)\sinh(yt)}{\sinh y}. \tag{47}$$

Since $y > 0$ and $0 \le t < 1$ we have $\cosh(ty) < \cosh y$ and $\sinh(ty) < \sinh y$ which implies $|g(t)| \le 1$. Then we have $|\int_{[0,1)} g(t)d\tau(t)| < 1$ which contradicts (46).

If $\widehat{\tau}(x) = 0$ and $\frac{d}{dx}\widehat{\tau}(x) = 0$ we have $\cos x + \int_{[0,1)} \cos(xt)d\tau(t) = 0$ and $\sin x + \int_{[0,1)} t\sin(xt)d\tau(t) = 0$. We proceed as above to reach a contradiction. A similar argument yields $|\frac{d}{dx}\widehat{\tau}(x)| + |\widehat{\tau}(x)| \ge c > 0$ on the real line. Here is an example. Let $r \ge 0$ and $\tau = \delta_{-1} + \delta_1 - r\chi_{[-1,1]}$ where χ_A denotes the indicator function of the set A. Then if $0 \le r < 1$ Lemma 12 implies that τ is mild while τ is not mild if $r > 1$. Indeed, $\widehat{\tau}(iy) = 2\cosh - 2r\sinh y/y$ and the equation $\cosh - r\sinh y/y = 0$ has a solution $y > 0$.

For a Radon measure ν the two following properties are equivalent: (a) ν is a solution of the convolution equation

$$\nu * \tau = 0 \tag{48}$$

and (b) for every compactly supported continuous function g the function $f = \nu * g$ belongs to $\mathcal{C}_\tau$.

Theorem 4. *Let τ be a mild measure and let Λ be defined by* (42)*. Then any Radon measure ν satisfying* (48) *is a tempered distribution and its distributional Fourier transform $\mu = \widehat{\nu}$ is a linear combination of Dirac measures supported by Λ.*

Theorem 4 is implicitly contained in the results obtained by Jean Delsarte, Jean-Pierre Kahane, and Bernard Malgrange on the theory of mean-periodic functions. On the one hand, Delsarte, Kahane, and Malgrange proved that when all the zeros of (42) are simple, the vector space $\mathcal{V}_\Lambda$ consisting of all linear combinations of the functions $\exp(i\lambda x)$, $\lambda \in \Lambda$, is dense in $\mathcal{C}_\tau$ for the topology of uniform convergence on compact intervals. On the other hand, when Λ is a uniformly discrete set of real numbers, trigonometric sums $g(x) \in \mathcal{V}_\Lambda$ satisfy uniform local L^2 estimates. More precisely there exist an interval $[0, T]$ and a constant $C \geq 1$ such that for any trigonometric sum $g(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(i\lambda x)$ we have

$$\Big(\int_y^{y+T} |g(x)|^2\,dx\Big)^{1/2} \leq C\Big(\int_0^T |g(x)|^2\,dx\Big)^{1/2} \tag{49}$$

uniformly in y. If ν is a solution of (48) we consider the convolution product $f = \nu * \phi$ where ϕ is an arbitrary compactly supported continuous function. We obviously have $f \in \mathcal{C}_\tau$ since $\nu * \tau * \phi = 0$. There exists a sequence f_j, $j \geq 1$, of trigonometric sums belonging to $\mathcal{V}_\Lambda$ and converging to f uniformly on compact intervals. Then (49) holds for f_j and passing to the limit it also holds for f. Therefore, $(\int_y^{y+T} |f(x)|^2\,dx)^{1/2}$ is uniformly bounded on the real line. Hence ν is a tempered distribution. The second assertion in Theorem 4 is then easy. We have $\widehat{\tau}\widehat{\nu} = 0$ and since the zeros of $\widehat{\tau}$ are simple, $\widehat{\nu}$ is a linear combination of Dirac measures on Λ. The proof of Theorem 4 is completed and we are now close to the construction of crystalline measures supported by uniformly discrete sets.

Theorem 5. *Let $a_1 < a_2 < \cdots < a_N$ be N real numbers and let c_j, $1 \leq j \leq N$, be some real or complex coefficients. We assume $c_1 \neq 0$ and $c_N \neq 0$. Let $\tau = \sum_1^N c_j\delta_{a_j}$. For any $a \in (a_1, a_N)$ there exists a unique Radon measure ν such that (i) ν is a solution of* (48) *and (ii) the restriction of ν to $(a_1, a_N]$ is δ_a. Moreover ν an atomic measure supported by a locally finite set. Finally, if τ is mild, ν is a tempered distribution and its distributional Fourier transform μ is a linear combination of Dirac measures supported by the Delone set Λ defined by* (42)*.*

Olevskii and Ulanovskii proved Theorem 5 independently with completely distinct methods [16]. In a variant on Theorem 5, τ is not a mild measure but satisfies a weaker hypothesis. More precisely in this new theorem one only assumes that the zeros of the Fourier–Laplace transform of τ are real and simple. Then the measure ν is a crystalline measure. This will be proved in a forthcoming paper. The proof of Theorem 5 is given in Section 7.

7. A proof of Theorem 5

The proof of Theorem 5 is detailed on a familiar example. From now on we focus on the measure

$$\tau_\alpha = \delta_0 - (1/2)\delta_1 - (1/2)\delta_\alpha + \delta_{1+\alpha} \tag{50}$$

where $\alpha > 1, \alpha \notin \mathbb{Q}$. A Radon measure ν belongs to $\mathcal{M}(\tau_\alpha)$ if $\nu * \tau_\alpha = 0$.

Lemma 13. *The measure τ_α is mild and we have $|\frac{d}{dx}\widehat{\tau}_\alpha(x)| \geq \frac{1+\alpha}{2}$ on the real line.*

We have $\tau_\alpha = \widetilde{\tau_\alpha} * \delta_{\frac{1+\alpha}{2}}$ where $\widetilde{\tau_\alpha} = \delta_{-\frac{1+\alpha}{2}} - (1/2)\delta_{\frac{1-\alpha}{2}} - (1/2)\delta_{\frac{-1+\alpha}{2}} + \delta_{\frac{1+\alpha}{2}}$ is an even measure. Then Lemma 12 can be applied to $\widetilde{\tau_\alpha}$ which implies Lemma 13.

Lemma 14. *Let $M = \{m + n\alpha; m \geq 0, n \geq 0\}$ be the additive semi-group generated by $0, 1$, and α. Then for any $a \in (0, 1+\alpha]$ there exists a unique atomic measure $\nu \in \mathcal{M}(\tau_\alpha)$ whose restriction to $(0, 1+\alpha]$ is δ_a. The support of ν is $(a + M) \cup (a - M)$.*

Indeed, if f is a function or a distribution then $f * \tau_\alpha = 0$ reads

$$f(x) = (1/2)f(x-1) + (1/2)f(x-\alpha) - f(x-\alpha-1). \tag{51}$$

Then (51) can be viewed as an evolution equation where x is the time variable. The initial condition is the restriction of f to $I = (0, 1+\alpha]$. If $1 + \alpha < x \leq 2 + \alpha$, we have $x - 1 \in I$, $x - \alpha \in I, x - \alpha - 1 \in I$, and (51) yields $f(x)$. If the restriction of f to $J_m = (0, m+\alpha]$, $m \in \mathbb{N}$, is a linear combination of Dirac measures then the same is true for the restriction of f to J_{m+1}. The same induction is applied to $(-\infty, 0)$ which ends the proof. The second assertion of Theorem 5 is implied by Theorem 4.

8. Explanation

The following theorem explains the strategy used in our constructions.

Theorem 6. *If the support Λ of a crystalline measure μ is uniformly discrete, then the distributional Fourier transform $\widehat{\mu}$ of μ is a mean-periodic measure.*

The proof is immediate. If Λ is a uniformly discrete set of real numbers there exists a non trivial compactly supported continuous function h such that $\widehat{h} = 0$ on Λ. We set $\widetilde{h}(x) = h(-x)$. Then $\widehat{h}\mu = 0$ which implies $\widetilde{h} * \widehat{\mu} = 0$ as announced. The simplest mean-periodic measures ν are the solutions of an equation $\nu * \tau = 0$ where τ is a finite linear combination of Dirac measures. This was our choice. Theorem 6 is completed by the following observation:

Lemma 15. *Let $\Lambda \subset \mathbb{R}$ a locally finite set. Let us assume that $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a crystalline measure. Then μ is an almost periodic measure and $\widehat{\mu}$ is a mean-periodic measure.*

Olevskii and Ulanovskii proved a deeper result in [17]. Under a technical hypothesis which is satisfied in all the existing examples they prove that there exists a finitely supported atomic measure τ such that $\tau * \widehat{\mu} = 0$. Let us prove Lemma 15. We denote by ϕ a positive even function in the Schwartz class whose Fourier transform $\widehat{\phi}$ is compactly supported. Let $\phi_u(x) = \phi(x-u)$. We then have $I(u) = \int \phi_u d\mu = (1/2\pi)\int \exp(-iuy)\widehat{\phi}(y)d\widehat{\mu}(y)$. Since $\widehat{\mu}$ is a Radon measure, $|I(u)| \leq C$. It implies that $\int_{u-1}^{u+1} d\mu \leq C$ and μ is translation bounded. Hence Λ is a finite union of uniformly discrete sets.

Let us prove that μ is an almost periodic measure. Since μ is translation bounded it suffices to prove that $\mu * g$ is an almost periodic function when g belongs to the

Schwartz class and $\widehat{g}$ is compactly supported. But this is trivial since $\mu * g$ is a finite trigonometric sum. Indeed the Fourier transform of $\mu * g$ is a finite linear combination of Dirac measures.

If $\Lambda \subset \mathbb{R}$ is a uniformly discrete set there exists a compactly supported continuous function h such that $\widehat{h} = 0$ on Λ. If $\Lambda = \cup_1^N M_j$ where M_j is uniformly discrete there exist N compactly supported continuous functions h_j such that $\widehat{h_j} = 0$ on M_j. We then observe that $h = h_1 * \cdots * h_N$ is not identically 0 since the product of two entire functions cannot be identically 0 unless one of them is identically 0. Then $\widehat{h} = 0$ on Λ. We now return to the crystalline measure μ. We set $\widetilde{h}(x) = h(-x)$. Then $\widehat{h}\mu = 0$ which implies $\widetilde{h} * \widehat{\mu} = 0$ as announced.

Lemma 16. *The measure $\nu_0 = \widehat{\mu}_0$ of Section 4 is not translation bounded.*

Indeed, we have $\int_0^R d|\nu_0| \geq CR^{3/2}$ which implies our claim. To prove this estimate it suffices to show that $\sum_{|m|,|n| \leq R} |\gamma_0(m,n)| \geq cR^{3/2}$. But this is implied by the estimate $|\gamma_0(m,n)| \geq c|n|^{-1/2}$, $c > 0$, which is valid if $|m| \leq |n|/4$.

9. The last proof

Here is our fourth proof of the existence of sparse crystalline measures is given. It does not depend on the preceding results but follows directly from [11]. As it was already mentioned a Radon measure μ on the real line is almost periodic if for every compactly supported continuous function g the convolution product $\mu * g$ is a Bohr almost periodic function. Let $\lfloor x \rfloor$ be the integral part of x and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Let α be an irrational number and let

$$\lambda_k = k + \{\alpha k\},\ k \in \mathbb{Z}. \tag{52}$$

Then $\sigma_\alpha = \sum_{k \in \mathbb{Z}} \delta_{\lambda_k}$ is not an almost-periodic measure. However $\Lambda_\alpha = \{\lambda_k;\ k \in \mathbb{Z}\}$ is a model set [10] and a weighted version $\sigma_\omega = \sum_{k \in \mathbb{Z}} \omega_k \delta_{\lambda_k}$ of σ_α is an almost-periodic measure supported by Λ_α [10]. Here is a deeper result:

Proposition 2. *The set Λ_α of real numbers defined by* (52) *cannot contain the support of a crystalline measure.*

We argue by contradiction. First Λ_α is a universal set of stable interpolation [9]. Given a Riemann integrable compact set K whose measure $|K|$ exceeds 2π (the density of Λ_α is 1) the following property holds: for any square integrable sequence $c(\lambda)$, $\lambda \in \Lambda_\alpha$, there exists a square integrable function f supported by K whose Fourier transform coincides with $c(\lambda)$ on Λ_α. By duality this is equivalent to the following property. There exists a constant C such that

$$\sum_{\lambda \in \Lambda_\alpha} |c(\lambda)|^2 \leq C^2 \int_K |\sum_{\lambda \in \Lambda_\alpha} c(\lambda) \exp(i\lambda x)|^2\, dx \tag{53}$$

is satisfied for any sequence $c(\lambda) \in l^2(\Lambda_\alpha)$. Then (53) is also satisfied for any finite sum $\sum_{\lambda \in \Lambda_\alpha} c(\lambda) \exp(-i\lambda x)$ which is simply the complex conjugate of $\sum_{\lambda \in \Lambda_\alpha} \overline{c(\lambda)} \exp(i\lambda x)$. Assuming that there exists a crystalline measure μ supported by Λ_α, let us disprove

(53). Let S be the support of the Fourier transform of μ. Let K be a Riemann integrable set such that (i) $K \cap S = \emptyset$ and (ii) $|K| > 2\pi$. Such a compact set exists since S is locally finite. Let ϕ be an even function supported by $[-\epsilon, \epsilon]$ and belonging to the Schwartz class. Here $\epsilon > 0$ is small enough to insure $(K + [-\epsilon, \epsilon]) \cap S = \emptyset$. Let Φ be the Fourier transform of ϕ. The product $\Phi\mu = \sum_{\lambda \in \Lambda_\alpha} c(\lambda)\delta_\lambda$ is an atomic measure and Proposition 1 implies that its total mass is finite. One can easily fix ϕ so that $\Phi\mu$ is not identically 0. It suffices to have $\phi(x) = g(x/\epsilon)$ where g is an even function supported by $[-1, 1]$. If ϵ is small enough $\Phi\mu$ is not identically 0. The Fourier transform of $\Phi\mu$ is the series $2\pi\phi * \widehat{\mu} = \sum_{\lambda \in \Lambda_\alpha} c(\lambda) \exp(-i\lambda x)$. This series vanishes on K which contradicts (53). Proposition 2 is proved.

The situation changes dramatically if the sawtooth function $x \mapsto \{x\}$ is replaced by a smooth 1-periodic function θ. As it was already mentioned, a $1-$periodic continuous function θ belongs to the Wiener algebra if its Fourier series is absolutely convergent. Such a function θ is identified to a continuous function on $\mathbb{R}/\mathbb{Z}$. Let α be an irrational real number. Let $\lambda_k = k + \theta(\alpha k)$, $\Lambda_\theta = \{\lambda_k,\, k \in \mathbb{Z}\}$, and $\sigma_\theta = \sum_{k \in \mathbb{Z}} \delta_{\lambda_k}$. The set Λ_θ is symmetric with respect to 0 if and only if θ is an odd function. If $\theta(x) = c$ is a constant function we have $\Lambda_\theta = \mathbb{Z} + c$. From now on this trivial case is excluded and θ is not a constant.

Proposition 3. *Let θ be a $1-$periodic real valued function belonging to the Wiener algebra. Then σ_θ is an almost periodic measure and its distributional Fourier transform is the atomic measure $\widehat{\sigma}_\theta$ defined by*

$$\widehat{\sigma}_\theta = 2\pi \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \gamma(p, q + \alpha p)\delta_{2\pi(q+\alpha p)} \tag{54}$$

where

$$\gamma(p, q + \alpha p) = \int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(pu + (q + \alpha p)\theta(u))]\, du. \tag{55}$$

Proposition 3 is Proposition 6.2 of [11]. Proposition 3 provides us with a direct proof of the existence of sparse crystalline measures.

Theorem 7. *Let α be an irrational real number. There exists a 1-periodic real valued function $\theta = \theta_\alpha$ with the following properties:*

- (a) *the derivative θ' of θ satisfies $\|\theta'\|_\infty < |\alpha|^{-1}$;*
- (b) *θ is a $\mathcal{C}^\infty$ function;*
- (c) *θ is odd;*
- (d) *σ_θ is a crystalline measure and Λ_θ is a Delone set.*

Proving Theorem 7 amounts to computing the integral in the right-hand side of (55). A change of variables is needed to do it. The change of variables and the construction of θ are unveiled after providing the reader with some heuristics. As stated in Theorem 7 the 1-periodic function θ is smooth and shall satisfy $\|\theta'\|_\infty < |\alpha|^{-1}$. Therefore the change of variables $u + \alpha\theta(u) = x$ defines a smooth diffeomorphism $\Theta \colon \mathbb{R} \mapsto \mathbb{R}$ which commutes with integral translations. For any $k \in \mathbb{Z}$ and $u \in \mathbb{R}$ we have $\Theta(u + k) = \Theta(u) + k$. The inverse diffeomorphism Θ^{-1} also commutes with integral translations. Since $\alpha \neq 0$ one can divide by α and define a 1-periodic smooth function ψ by $\Theta^{-1}(x) =$

$x - \alpha\psi(x)$. This looks artificial but will be explained in a moment. We have $\theta(u) = \psi(x) = \psi(u + \alpha\theta(u))$ identically. Finally, ϕ is defined by $\phi(x) = \psi(x) + Nx = (N\alpha + 1)\theta(u) + Nu$ where N belongs to $\mathbb{N}$ and will be specified. For technical reasons α is negative in the proof of Theorem 7. Since θ is odd the sign of α is irrelevant.

From now on α is negative, an assumption which can be eventually dropped. The proof starts the other way around and begins with the definition of ϕ. Then θ will be obtained by inversing the preceding construction. We start with a Blaschke product B in the open unit disc $\mathbf{D}$. A Blaschke product is defined by a finite set $z_1, \ldots, z_N$, of complex numbers belonging to the open unit disc. We have $B(z) = \Pi_1^N \frac{z - z_j}{1 - z\bar{z}_j}$. The boundary $\mathbf{T}$ of $\mathbf{D}$ is now identified to $\mathbb{R}/\mathbb{Z}$ via $z = \exp(2\pi i x)$. Then the real valued function ϕ is defined as the phase of the restriction of $B(z)$ to $\mathbf{T}$. We have $\exp(2\pi i\phi) = B(\exp(2\pi i x))$ on $\mathbb{R}/\mathbb{Z}$. This phase is not continuous on $\mathbb{R}/\mathbb{Z}$ but there exists a 1-periodic smooth function ψ on $\mathbb{R}$ such that $\phi(x) = \psi(x) + Nx$ on $\mathbb{R}$. If $\|\psi'\|_\infty < |\alpha|^{-1}$ the map $x \mapsto u = x - \alpha\psi(x)$ is a diffeomorphism of $\mathbb{R}$ which commutes with integral translations. The inverse mapping also commutes with integral translations and is given by $x = u + \alpha\theta(u)$. This defines the 1-periodic smooth function θ we are looking for. Obviously θ depends on α while ψ does not.

We now return to the computation of

$$I(p, q) = \int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(pu + (q + \alpha p)\theta(u))]\, du. \tag{56}$$

Performing the change of variable $x = u + \alpha\theta(u)$ or $u = x - \alpha\psi(x)$ we obtain

$$I(p, q) = \int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(px + q\psi(x))](1 - \alpha\psi'(x))\, dx. \tag{57}$$

This can be written

$$I(p, q) = \int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i((p - qN)x + q\phi(x))](1 - \alpha\psi'(x))\, dx. \tag{58}$$

Our choice of ϕ implies that $I(p, q) = 0$ for many pairs (p, q). Since $B(x) = \exp(2\pi i\phi(x))$ is the trace on $\mathbf{T}$ of a holomorphic function in the disc $\mathbf{D}$ we have

$$\int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(mx + q\phi(x))]\, dx = 0 \tag{59}$$

if $q \le -1$ and $m \le -1$, or $q \ge 1$ and $m \ge 1$. On the other hand, (59) and an integration by parts yields

$$\int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(m \cdot x + q\phi(x))]\phi'(x)\, dx = 0 \tag{60}$$

if $q \le -1$ and $m \le -1$, or $q \ge 1$ and $m \ge 1$. Then (59) and (60) imply

$$\int_{\mathbb{R}/\mathbb{Z}} \exp[-2\pi i(m \cdot x + q\phi(x))](1 - \alpha\psi'(x))\, dx = 0 \tag{61}$$

if $q \le -1$ and $m \le -1$, or $q \ge 1$ and $m \ge 1$.

Therefore $I(p, q) = 0$ if $q \le -1$ and $p - Nq \le -1$ or $q \ge 1$ and $p - Nq \ge 1$. It follows that the spectrum of σ_θ is locally finite if $-1 < N\alpha < 0$.

We now choose N real numbers $z_j = r_j \in (-1,1)$ and consider the corresponding Blaschke product $B(z)$. Then $\overline{B(z)} = B(\overline{z})$ and ϕ is an odd function. We have $\phi(x) = \psi(x) + Nx$ and $\psi = \psi_1 + \cdots + \psi_N$ where $\psi_j(x)$ is the phase of $1 - r_j \exp(-2\pi x)$. It implies $\|\psi_j'\|_\infty \leq \frac{2\pi r_j}{1-r_j}$. If $\frac{2\pi r_1}{1-r_1} + \cdots + \frac{2\pi r_N}{1-r_N} < |\alpha|^{-1}$ the preceding scheme can be completed. Then θ is also an odd function and α can be replaced by $-\alpha$. Theorem 7 is fully proved.

Proposition 4. *The set Λ_θ of Theorem 7 is a curved model set.*

We have $\Lambda_\theta = \{\lambda_k,\, k \in \mathbb{Z}\}$ where $\lambda_k = k + \theta(\alpha k)$, $k \in \mathbb{Z}$. Let us prove that $(\alpha\lambda_k, \lambda_k)$ belongs, modulo $\mathbb{Z}^2$, to the graph $\Gamma \subset (\mathbb{R}/\mathbb{Z})^2$ of the function ψ which is defined in the proof of Theorem 7. Indeed we have $\alpha\lambda_k = \alpha k + \alpha\theta(\alpha k)$. If $u_k = \alpha k$ and $x_k = \alpha\lambda_k$ this can be written as $x_k = u_k + \alpha\theta(u_k)$. Therefore $u_k = x_k - \alpha\psi(x_k)$ and $(x_k, k + \theta(\alpha k))$ is congruent to $(x_k, \theta(u_k)) = (x_k, \psi(x_k))$ as stated. Conversely if $(\alpha\lambda, \lambda)$ belongs to Γ modulo $\mathbb{Z}^2$ we have $\psi(\alpha\lambda) = \lambda - k$ for some $k \in \mathbb{Z}$. It implies $\lambda = k + \theta(\alpha k)$.

10. A general Dirichlet series

In this section we follow [4] and consider general Dirichlet series of the form

$$F(s) = \sum_{m\geq 0}\sum_{n\geq 0} c(m,n)(m + n\sqrt{2})^{-s}. \tag{62}$$

One assumes that $c(0,0) = 0$ and that there exists a positive β such that

$$\sum_{m\geq 0}\sum_{n\geq 0} |c(m,n)|(m + n\sqrt{2})^{-\beta} < \infty. \tag{63}$$

This implies that $F(s)$ is holomorphic in $\Re s > \beta$. A first observation is given by the following lemma

Lemma 17. *The collection of holomorphic functions $F(s)$ defined by* (62) *and* (63) *is an algebra.*

This follows from the fact that $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is a ring. We now study specific examples. As we did in Section 4 we set $\widetilde{\gamma} = \gamma_1 + \gamma_2 - 2\gamma_0$ and consider the atomic measure

$$\nu = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \widetilde{\gamma}(m,n)\delta_{m+n\sqrt{2}}. \tag{64}$$

We have $\nu(\{0\}) = 0$. Moreover ν is an even measure and the distributional Fourier transform of ν is a translation bounded atomic measure supported by a uniformly discrete set Λ which does not contain 0.

Theorem 8. *With these notations*

$$\xi(s) = \sum_{m\geq 0}\sum_{n\geq 0} \widetilde{\gamma}(m,n)(m + n\sqrt{2})^{-s} \tag{65}$$

is the restriction to $\Re s > 2$ of an entire function of the complex variable s.

The series (65) converges if $\Re s > 2$ since $|\widetilde{\gamma}(m,n)| \leq C$. We now mimic an argument used in [4] and in [14]. Lemma 18 will provide us with an integral representation of $\xi(s)$ which implies that $\xi(s)$ is an entire function of s. For simplifying the notation let us order the real numbers $m + n\sqrt{2}, m \geq 0, n \geq 0, (m,n) \neq (0,0)$, as an increasing sequence $0 < s_1 < s_2 < \cdots$. Similarly we set $a_k = \widetilde{\gamma}(m,n)$ if $s_k = m + n\sqrt{2}$. We then have $\xi(s) = 2\sum_1^\infty a_k s_k^{-s}$. If $g_u(x) = \exp(-ux^2)$, $u > 0$ let us define

$$\theta(u) = \frac{1}{2}\int g_u \, d\nu = \sum_1^\infty a_k \exp(-us_k^2). \tag{66}$$

We then have

Lemma 18. *There exists a positive real number α such that*

$$\theta(u) = O(\exp(-\alpha u)),\ u \to +\infty \tag{67}$$

and

$$\theta(u) = O(\exp(-\alpha/u)),\ u > 0, u \to 0. \tag{68}$$

The first assertion is obvious since $|a_k| \leq 2$ and $s_1 > 0$. We prove the second claim with a smaller value of $\alpha > 0$. The Fourier transform of $g_u(x) = \exp(-ux^2)$, $u > 0$, is $\widehat{g}_u(y) = \sqrt{\pi}u^{-1/2}\exp(-y^2/4u)$. Then

$$\theta(u) = \langle g_u, \nu\rangle = (2\pi)^{-1}\langle \widehat{g}_u, \widehat{\nu}\rangle. \tag{69}$$

We know that $\mu = \widehat{\nu}$ is a translation bounded atomic measure supported by a uniformly discrete set Λ which does not contain 0. We have $\mu = \sum_{k=-\infty}^{\infty} c_k \delta_{\lambda_k}$ where $\lambda_{-k} = -\lambda_k$ and $0 < \lambda_1 < \cdots < \lambda_k < \cdots$. Moreover $c_{-k} = c_k$, $c_0 = 0$, and $|c_k| \leq C$. Therefore

$$\langle \widehat{g}_u, \mu\rangle = 2\sqrt{\pi}u^{-1/2}\sum_1^{+\infty} c_k \exp(-\lambda_k^2/4u). \tag{70}$$

This implies $|\langle \widehat{g}_u, \mu\rangle| = O(\exp(-\alpha/u))$, $u > 0, u \to 0$, which is (68).

Lemma 19. *The function ξ defined by* (65) *is the Mellin transform of θ. More precisely we have for $\Re s > 2$*

$$\xi(s) = \frac{2}{\Gamma(s/2)}\int_0^\infty \theta(t)\, t^{s/2}\, \frac{dt}{t}. \tag{71}$$

To prove Lemma 19 it suffices to exchange the summation and the integration in (71). Since $\Re s > 2$ this is guaranteed by Lemma 18. But (67) and (68) imply that the right-hand side of (71) is an entire function of s. This ends the proof of Theorem 8. We now prove a functional equation of the type studied by Kahane and Mandelbrojt in [4].

Let us consider $\xi_1(s) = 2\sum_1^\infty c_k \lambda_k^{-s}$ and $\theta_1(u) = \sum_1^\infty c_k \exp(-u\lambda_k^2)$. We have $\xi_1(s) = \frac{1}{2\Gamma(s/2)}\int_0^\infty \theta_1(t)\, t^{s/2}\, \frac{dt}{t}$. But $\theta_1(t) = \frac{\sqrt{\pi}}{2}t^{-1/2}\theta(1/4t)$ which yields $\Gamma(s/2)\xi_1(s) = 2\sqrt{\pi}\,\Gamma(\frac{1-s}{2})\xi(1-s)$. Up to the normalization of the Fourier transform this is the Kahane–Mandelbrojt identity.

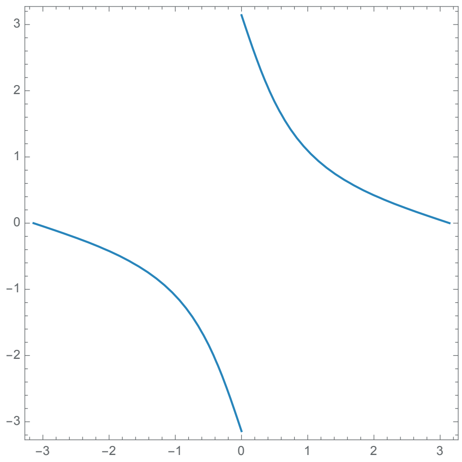

Figure 1. The curve $\tan(x/2)\tan(y/2) = 1/3$.

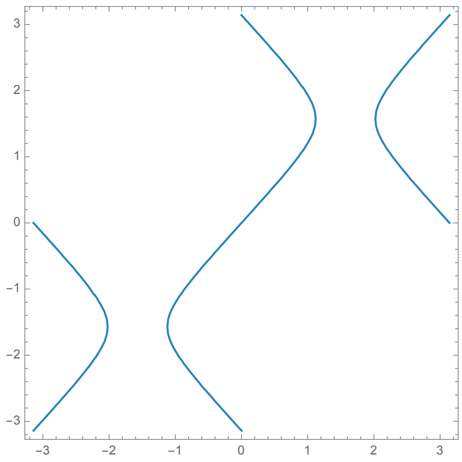

Figure 2. The curve $\sin x = r \sin y$, $r = 0.9$.

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