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FROM SALOMON BOCHNER TO DAN SHECHTMAN

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Yves François Meyer

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Co-editor

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Elvegata 17, 7012 Trondheim

e-mail: post@dknvs.no
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Yves François Meyer

1. Introduction

Sixty years ago Jean-Pierre Kahane made some seminal contributions to the theory of mean-periodic functions. His lectures at the Tata Institute (1957) are a remarkable source of exciting problems [8]. Two years later Kahane investigated the structure of some sets of frequencies $\Lambda \subset \mathbb{R}^n$ which delimit the boundary between mean-periodic and almost-periodic functions and defined "property $Q(\Lambda)$" by the following condition: *Any mean periodic function f whose spectrum is simple and contained in Λ is a Bohr almost periodic function* [9]. Ten years later it was observed that Kahane's property $Q(\Lambda)$ is also seminal in the problem of spectral synthesis [17]. The sets Λ which satisfy $Q(\Lambda)$ are named *coherent sets of frequencies* in [17].

Twenty years before Kahane's seminal paper was published, Salomon Bochner characterized the Fourier–Stieltjes transforms of bounded Radon measures [2]. A closed set set $\Lambda \subset \mathbb{R}^n$ satisfies *Bochner's property* if this characterization is still valid for the restrictions of these Fourier–Stieltjes transforms to Λ. *Kahane's property $Q(\Lambda)$ is equivalent to Bochner's property if Λ is a locally finite set.* This unexpected fact is proved in this note. One implication is trivial since $Q(\Lambda)$ obviously implies Bochner's property. The proof of the converse implication paves the road to (a) a definition of the *harmonic coherence score* of a set $\Lambda \subset \mathbb{R}^n$ and (b) a new approach to the mathematical theory of

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quasi-crystals. Let us explain this odd remark. In 1982 Dan Shechtman discovered that quasi-crystals exist in the surrounding world [23]. It is shown in this note that *weak characters* pave the road which goes from Bochner to Shechtman. On the way we are visiting *coherent sets of frequencies, harmonious sets*, we are studying the fascinating problem of extensions of positive definite functions, we are proving the main theorem of this note, and finally we arrive at *model sets*. That is why at the end of the journey we are reaching Shechtman's quasi-crystals. Indeed Michel Duneau, Denis Gratias, André Katz, and Robert Moody discovered that the quasi-crystals elaborated by Dan Shechtman can be modeled by *model sets*. On this tour we are also admiring some Penrose's pavings [1] since the set of vertices of most of the Penrose pavings are *model sets*, as it was proved by N. G. de Bruijn in [3].

This paper is almost an autobiography since large pieces of my early work are revisited and better understood. On the way from Bochner to Shechtman we meet Michael Baake, Denis Gratias, Stanislaw Hartman, Jean-Pierre Kahane, Yitzhak Katznelson, Jeffrey Lagarias, Jean-François Méla, Robert Moody, Alexander Olevksii, Haskell Rosenthal, Walter Rudin, and Nicholas Varopoulos who are or were my colleagues and my friends. I wish to express my sincere gratitude for all they gave me.

This paper is organized as follows: Bochner's property and coherent sets of frequencies are defined in Sections 2 and 3. A locally finite set $\Lambda \subset \mathbb{R}^n$ satisfies Bochner's property if and only if Λ is a coherent set of frequencies. This is Theorem 3.1 completed by Theorem 4.2 and its proof is given in Sections 4 and 5. Weak characters are defined in Section 4. They play a key role in the proof of Theorem 3.1. Theorem 3.1 is applied to harmonious sets and to the Pisot set in Section 6. Harmonious sets open the door to the mathematical theory of quasi-crystals as explained in Section 6. Some complementary results are given in Section 7.

2. Two theorems by Bochner

Let us fix some notations. Functions are real or complex valued. The Lebesgue space $L^p(\mathbb{R}^n)$ is equipped with the L^p norm defined by $\|f\|_p = (\int_{\mathbb{R}^n} |f|^p\, dx)^{1/p}$ if $1 \le p < \infty$ and $\|f\|_\infty = \sup \mathrm{ess}_{x\in\mathbb{R}^n} |f(x)|$. The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is the continuous function on $\mathbb{R}^n$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x\cdot\xi) f(x)\, dx. \tag{1}$$

Similarly the Fourier–Stieltjes transform of a bounded Radon measure μ is the continuous function on $\mathbb{R}^n$ defined by $\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x\cdot\xi)\, d\mu(x)$.

A complex valued function ϕ of the real variable x is positive definite if and only if for any N and any $x_j \in \mathbb{R}$, $c_j \in \mathbb{C}$, $1 \le j \le N$, we have

$$\sum_1^N \sum_1^N c_j \bar{c}_k \phi(x_j - x_k) \ge 0. \tag{2}$$

This definition was proposed by M. Mathias in [15]. Nine years later (1932) Bochner proved the following:

Theorem 2.1. *If ϕ is a positive definite function of the real variable x, if $\phi(0) = 1$ and if ϕ is continuous at 0 then ϕ is the Fourier–Stieltjes transform of a probability measure μ.*

Probabilists call ϕ the *characteristic function* of the probability measure μ [5]. If a sequence ϕ_j, $j = 1, 2, \ldots$ of characteristic functions converge pointwise to a function g and if g is continuous at 0, then g is also a characteristic function. Indeed only finite sets are involved in (2). This fact is seminal for understanding the convergence in law of a sequence of random variables.

Mark G. Krein, Walter Rudin, and Palle E.T. Jorgensen [7], [11], [21], raised some fascinating issues on restrictions and extensions of continuous positive definite functions. Let $E \subset \mathbb{R}^n$ be a set and $\Lambda = E - E$ the set of all differences $x - y$, $x, y \in E$. A function ϕ defined on Λ is positive definite if for any N and any $x_j \in E$, $c_j \in \mathbb{C}$, $1 \le j \le N$, we have

$$\sum_1^N \sum_1^N c_j \bar{c}_k \phi(x_j - x_k) \ge 0. \tag{3}$$

We say that the set Λ satisfies property $\mathcal{R}$ if any continuous positive definite function ϕ on Λ is the restriction to Λ of a continuous positive definite function Φ on $\mathbb{R}^n$. This problem will be elucidated in Section 4 when Λ is a locally finite set. Locally finite sets are defined as follows:

Definition 2.1. *A set Λ is locally finite if for any compact set K the intersection $K \cap \Lambda$ is a finite set.*

We now consider Fourier–Stieltjes transforms of complex valued Radon measures in n dimensions. Bounded Radon measures are signed or complex valued measures. The total mass of such a measure μ is denoted by $\|\mu\|$. The Banach space $\mathcal{B}$ of bounded Radon measures, equipped with the norm $\|\mu\|$, is the dual of the space $\mathcal{C}_0$ of all continuous functions on $\mathbb{R}^n$ tending to 0 at infinity. The norm of $f \in \mathcal{C}_0$ is the sup norm $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$ and the total mass $\|\mu\|$ of the Radon measure μ is the norm of the linear functional on $\mathcal{C}_0$ defined by $f \mapsto \int f\, d\mu$.

In 1934 Bochner characterized the Fourier–Stieltjes transforms of bounded Radon measures by the following property:

Theorem 2.2. *The following two properties of a function ϕ defined on $\mathbb{R}^n$ are equivalent:*

1. *ϕ is the Fourier–Stieltjes transform of a bounded Radon measure μ.*
2. *ϕ is continuous on $\mathbb{R}^n$ and there exists a constant C such that for any finitely supported measure σ on $\mathbb{R}^n$ one has*

$$\left| \int \phi \, d\sigma \right| \le C \|\widehat{\sigma}\|_\infty. \tag{4}$$

One way is obvious. Indeed if μ is a finite Radon measure and if $\phi = \widehat{\mu}$ we have $|\int \phi\, d\sigma| = |\int \widehat{\sigma}\, d\mu| \le \|\widehat{\sigma}\|_\infty \|\mu\|$. Theorem 2.2 would be trivial if (4) was replaced by $|\int \phi(x) f(x)\, dx| \le C \|\widehat{f}\|_\infty$ for f in L^1. Then $\widehat{f}$ tends to 0 at infinity and $\widehat{\phi}$ can be extended to a continuous linear form on $\mathcal{C}_0$. By the Riesz–Markov–Kakutani representation theorem this linear form is given by a bounded Radon measure μ whose Fourier transform is $\tilde{\phi}$ where $\tilde{\phi}(x) = \phi(-x)$.

Property (4) can be written explicitly: There exists a constant C such that for any integer N, any coefficients $c_1, \dots, c_n \in \mathbb{C}$, and any points $x_1, \dots, x_N \in \mathbb{R}^n$, one has $|\sum_1^N c_k \phi(x_k)| \le C \,\|\sum_1^N c_k \exp(2\pi i\, x_k \cdot x)\|_\infty$. Indeed the Fourier transform of the measure $\sigma = \sum_1^N c_k \delta_{x_k}$ is the trigonometric sum $P(x) = \sum_1^N c_k \exp(-2\pi i\, x_k \cdot x)$.

The proof of Theorem 2.2 given by Rudin in [22, p. 32], relies on the theory of almost periodic functions which is summarized in the following lines. A set $M \subset \mathbb{R}^n$ is relatively dense if there exists a compact set $K \subset \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ the intersection $(x+K) \cap M$ is not empty. This is equivalent to $M - K = \mathbb{R}^n$. A bounded and continuous function f on $\mathbb{R}^n$ is almost periodic if and only if for any positive ϵ the set M_ϵ of ϵ almost periods of f is relatively dense. An ϵ almost period τ of f is defined by $\sup_x |f(x+\tau) - f(x)| \le \epsilon \|f\|_\infty$. A trigonometric sum is a finite sum $P(x) = \sum_{\omega \in S} c(\omega) \exp(2\pi i\, \omega \cdot x)$ where S, the spectrum of P, is a finite subset of $\mathbb{R}^n$. Any trigonometric sum is almost periodic. Conversely an almost periodic function f is a uniform limit of a sequence P_j of trigonometric sums. We have $\lim_{j\to\infty} \|f - P_j\|_\infty = 0$. In the framework of Pontryagin duality the Bohr compactification $\mathcal{G}$ of $\mathbb{R}^n$ is the dual group of $\mathbb{R}^n$ equipped with the discrete topology. The subgroup $\mathbb{R}^n$ of $\mathcal{G}$ is dense in $\mathcal{G}$. Finally any almost periodic function f is the restriction to $\mathbb{R}^n$ of a continuous function F on $\mathcal{G}$. Conversely if F is continuous on $\mathcal{G}$ its restriction to $\mathbb{R}^n$ is almost periodic. A simplified version of the group $\mathcal{G}$ will be used in Section 4.

Rudin in [22] proved that Theorem 2.2 can be deduced from Theorem 2.1. This beautiful proof is our guide in this note and is given now. If ϕ satisfies (4) one denotes by $\mathcal{L}_\phi$ the linear form defined by $\mathcal{L}_\phi(P) = \sum c(\lambda)\phi(\lambda)$ when $P = \sum c(\lambda) \exp(2\pi i \lambda \cdot x)$. We have by (4) $|\mathcal{L}_\phi(P)| \le C\|P\|_\infty$. Therefore $\mathcal{L}_\phi$ extends to a continuous linear form on the Banach space of almost periodic functions. These almost periodic functions are the continuous functions on $\mathcal{G}$. By the Riesz–Markov–Kakutani representation theorem there exists a Radon measure μ on $\mathcal{G}$ such that $\mathcal{L}_\phi(P) = \int_{\mathcal{G}} P\, d\mu$. Therefore $\tilde{\phi}$ which is defined by $\tilde{\phi}(x) = \phi(-x)$ is the Fourier–Stieltjes transform of μ. To conclude the proof it suffices to show that μ is in fact carried by $\mathbb{R}^n$. We observe that there exists a Borel function ξ on $\mathcal{G}$ such that $|\xi| = 1$ everywhere on $\mathcal{G}$ and $\xi\mu = |\mu|$. Since continuous functions are dense in $L^1(\mathcal{G}, d|\mu|)$ there exists a sequence P_j of trigonometric sums such that $\int_{\mathcal{G}} |\xi - P_j|\, d|\mu| \to 0$ as $j \to \infty$. We set $\mu_j = P_j\mu$. Then the measure $|\mu| = \xi\mu$ is the limit in norm of the sequence of measures $\mu_j = P_j\mu$ as $j \to \infty$. Let ψ be the Fourier transform of $|\mu|$. By definition ψ is positive definite. Let us prove that ψ is a continuous function. On the one hand $\widehat{\mu}_j = \phi_j$ converges uniformly to ψ on $\mathbb{R}^n$. On the other hand $\widehat{\mu}_j = \phi_j$ is a linear combination of translates of ϕ. Therefore ϕ_j is a continuous function and so is ψ. The positive definite function ψ is continuous and, by Theorem 2.1, is the Fourier transform of a positive measure ν on $\mathbb{R}^n$. But $\nu = |\mu|$ since they have the same Fourier transform. Therefore μ is carried by $\mathbb{R}^n$ as announced.

Two restriction algebras are defined now. They are needed to define Bochner's property (Definition 2.4). One shall begin with the Wiener algebra. A function f on

$\mathbb{R}^n$ belongs to the Wiener algebra $A(\mathbb{R}^n)$ if f is the Fourier transform of an integrable function F: $f = \widehat{F},\ F \in L^1(\mathbb{R}^n)$. The norm $\|f\|_A$ of f in $A(\mathbb{R}^n)$ is $\|F\|_1$ by definition. Any function $f \in A(\mathbb{R}^n)$ is continuous on $\mathbb{R}^n$ and tends to 0 at infinity. We have $A(\mathbb{R}^n) \subset \mathcal{C}_0(\mathbb{R}^n)$. The product w between two functions u and v in $A(\mathbb{R}^n)$ is the pointwise product $w(x) = u(x)v(x)$. Then $A(\mathbb{R}^n)$ is a Banach algebra [10], [22]. The space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions is dense in $A(\mathbb{R}^n)$. The dual space of $A(\mathbb{R}^n)$ is the space $PM(\mathbb{R}^n)$ of "pseudo-measures". A pseudo-measure is a tempered distribution S whose distributional Fourier transform belongs to $L^\infty(\mathbb{R}^n)$. A bounded Radon measure is obviously a pseudo-measure. In one dimension the tempered distribution p.v. $1/x$ is a pseudo-measure which is not a measure.

Similarly a continuous function f on $\mathbb{R}^n$ belongs to $B(\mathbb{R}^n)$ if f is the Fourier–Stieltjes transform of a bounded Radon measures μ on $\mathbb{R}^n$. The norm of $f = \widehat{\mu}$ in $B(\mathbb{R}^n)$ is the total mass $\|\mu\|$ of μ. The product between two functions in $B(\mathbb{R}^n)$ is given by the pointwise multiplication. If $f \in B(\mathbb{R}^n)$ and $u \in A(\mathbb{R}^n)$ then $v = fu$ belongs to $A(\mathbb{R}^n)$. Indeed if μ is a bounded Radon measure and $g \in L^1$ then the convolution product $\mu * g$ belongs to L^1.

S-E. Takahasi and O. Hatori observed that Theorem 2.2 makes sense for Banach algebras [25]. We describe their work in a slightly simplified version which suffices in what follows. Let $\Lambda \subset \mathbb{R}^n$ be a closed set. The Banach algebra of all continuous functions on Λ tending to 0 at infinity is denoted by $\mathcal{C}_0(\Lambda)$. Let A be a Banach algebra contained in $\mathcal{C}_0(\Lambda)$. We assume that Λ is the Gelfand spectrum of A and that $\|f\|_\infty \le \|f\|_A$ for any $f \in A$. Let A^* be the dual vector space of the Banach space A. Then any bounded Radon measure μ supported by Λ defines a linear form on A by $f \mapsto \int_\Lambda f\,d\mu$. Therefore $\mu \in A^*$. The following definition was proposed by Takahasi and Hatori [25]:

Definition 2.2. *The Banach algebra A satisfies the Bochner–Schoenberg–Eberlein's property if the following condition is satisfied: Let f be a continuous function on Λ. Let us assume that there exists a constant C such that for any atomic measure σ supported by Λ we have:*

$$\left|\int_\Lambda f\,d\sigma\right| \le C\|\sigma\|_{A^*}. \tag{5}$$

Then f is a multiplier of the Banach algebra A.

This definition is seminal in our work. It exemplifies the key role which is played by multipliers in Bochner's property. A multiplier f of A is a continuous function on Λ such that for any $u \in A$ the pointwise product fu still belongs to A. Let us give a simple example of the Bochner–Schoenberg–Eberlein's property. Let $A = A(\mathbb{R}^n)$ be the Wiener algebra. The dual space A^* is then the space $PM(\mathbb{R}^n)$ of pseudo-measures. Then we have:

Proposition 2.1. *The Wiener algebra $A(\mathbb{R}^n)$ satisfies the Bochner–Schoenberg–Eberlein's property.*

Proposition 2.1 is identical to Theorem 2.2. Indeed a multiplier of $A(\mathbb{R}^n)$ is a function of $B(\mathbb{R}^n)$.

Our goal is to extend Theorem 2.2 to some restriction algebras which are defined now. Let $\Lambda \subset \mathbb{R}^n$ be a closed set. The restriction algebra $A(\Lambda)$ consists of the restrictions to Λ of the functions of the Wiener algebra $A(\mathbb{R}^n)$. These restrictions are well defined since any $f \in A(\mathbb{R}^n)$ is a continuous function. The product w between two functions u and v in $A(\Lambda)$ is given by the pointwise multiplication: $w(\lambda) = u(\lambda)v(\lambda)$, $\lambda \in \Lambda$. The norm of $(f(\lambda))_{\lambda\in\Lambda}$ in $A(\Lambda)$ is the quotient norm. More precisely

$$\|f\|_{A(\Lambda)} = \inf\{\|F\|_1;\ \widehat{F} = f \text{ on } \Lambda\}. \tag{6}$$

Let $I(\Lambda)$ be the closed ideal of $A(\Lambda)$ consisting of all the functions $f \in A(\mathbb{R}^n)$ which vanish on Λ. The Banach algebra $A(\Lambda)$ is the quotient algebra $A(\mathbb{R}^n)/I(\Lambda)$. The dual space of $A(\Lambda)$ is the annihilator of $I(\Lambda)$ in $PM(\mathbb{R}^n)$. In other words it is the space of pseudo-measures S supported by Λ which satisfy the property $< S, f >= 0$ for any $f \in I(\Lambda)$. If this is satisfied for any pseudo-measure S supported by Λ we say that Λ is a set of spectral synthesis [10], [22]. In Theorem 3.1 Λ is a locally finite set. Therefore it is a set of spectral synthesis and the dual space of $A(\Lambda)$ is the space $PM(\Lambda)$ of pseudo-measures supported by Λ. Then the inverse Fourier transform of any $S \in PM(\Lambda)$ is a function $f \in L^\infty(\mathbb{R}^n)$ and f is the sum of a trigonometric series $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\, x\cdot\lambda)$ whose frequencies belong to Λ. If S is an atomic measure supported by Λ its inverse Fourier transform is an almost-periodic function whose frequencies belong to Λ.

Similarly $B(\Lambda)$ denotes the Banach algebra of the restrictions to Λ of the functions in $B(\mathbb{R}^n)$. Here again the product of two functions in $B(\Lambda)$ is given by the pointwise multiplication. The norm in $B(\Lambda)$ of $f \in B(\Lambda)$ is the quotient norm, defined as the lower bound of $\|\mu\|$ computed on the bounded measures μ satisfying $\widehat{\mu}(\lambda) = f(\lambda)$, $\forall\lambda \in \Lambda$. We obviously have $A(\Lambda) \subset B(\Lambda)$. These definitions and notations can be found in [10] or [22].

Definition 2.3. *A closed set $\Lambda \subset \mathbb{R}^n$ satisfies Bochner's property if for any continuous function ϕ defined on Λ the following two properties are equivalent*

1. $\phi \in B(\Lambda)$.
2. *There exists a constant C such that* (4) *is satisfied for any finitely supported measure σ whose support is a contained in Λ.*

The following definition eases our understanding of Bochner's property.

Definition 2.4. *Let $\Lambda \subset \mathbb{R}^n$ be a closed set. We denote by $\mathcal{M}(\Lambda)$ the Banach space consisting of all continuous functions ϕ on Λ such that*

$$\left|\int \phi\, d\sigma\right| \le C\|\widehat{\sigma}\|_\infty \tag{7}$$

is satisfied for a constant C and any finite linear combination $\sigma = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ of Dirac measures supported by Λ. The norm $\|\phi\|_{\mathcal{M}(\Lambda)}$ of $\phi \in \mathcal{M}(\Lambda)$ is the lower bound of these constants C.

We have $\|\phi\|_\infty \le \|\phi\|_{\mathcal{M}(\Lambda)}$. This is implied by (7) when σ is a Dirac measure. We obviously have $B(\Lambda) \subset \mathcal{M}(\Lambda)$ and $|\phi\|_{\mathcal{M}(\Lambda)} \le |\phi\|_{B(\Lambda)}$.

Here is a second definition of Bochner's property.

Definition 2.5. *A closed set $\Lambda \subset \mathbb{R}^n$ satisfies Bochner's property if $B(\Lambda) = \mathcal{M}(\Lambda)$.*

If Λ satisfies Bochner's property the closed graph theorem implies the existence of a constant C_0 such that for any $\phi \in B(\Lambda)$ we have $\|\phi\|_{B(\Lambda)} \le C_0 \|\phi\|_{\mathcal{M}(\Lambda)}$. But $\|\phi\|_{B(\Lambda)} \ge \|\phi\|_{\mathcal{M}(\Lambda)}$ for any ϕ which implies $C_0 \ge 1$. The case $C_0 = 1$ is of special interest. Then $B(\Lambda) = \mathcal{M}(\Lambda)$ is an isometry.

Definition 2.6. *The harmonic coherence score of Λ is the lower bound of these constants C_0 and is denoted by $\omega(\Lambda)$.*

We always have $\omega(\Lambda) \ge 1$. In this note we focus on the case of locally finite sets Λ. Then $\Lambda_1 \subset \Lambda_2$ implies $\omega(\Lambda_1) \le \omega(\Lambda_2)$. This is not true in general. For instance $\omega(\mathbb{R}) = 1$ but there exist many closed sets Λ of real numbers for which $\omega(\Lambda) = \infty$. The simplest example is $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$ when α is irrational. The harmonic coherence score of a finite set is 1. Therefore we do not have $\omega(\Lambda) = \sup_{F \subset \Lambda} \omega(F)$ where this upper bound is computed on the collection of all finite subsets of Λ. When Λ is a harmonious set (Definition 6.1) we have $\omega(\Lambda) = 1$. Is it a characterization of harmonious sets? Can $\omega(\Lambda)$ be arbitrarily large when $\Lambda \subset \mathbb{R}$? An example is constructed (Theorem 4.3) where $\Lambda \subset \mathbb{R}$ and $\omega(\Lambda) = \theta > 1$. Then if $\Lambda^n = \Lambda \times \cdots \times \Lambda$ (n times) we have $\omega(\Lambda^n) = \theta^n$.

Proposition 2.2. *The Banach space $\mathcal{M}(\Lambda)$ is a Banach algebra contained in the algebra of continuous functions on Λ.*

Let $\mathcal{T}_\Lambda$ be the vector space consisting of all finite trigonometric sums $P(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\, x \cdot \lambda)$ whose frequencies belong to Λ. Then (7) can be written as $|\sum_{\lambda\in\Lambda} c(\lambda)\phi(\lambda)| \le C\|\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda \cdot x)\|_\infty$ for any $P \in \mathcal{T}_\Lambda$. This applies to $P(x+x_0) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x_0)\exp(2\pi i\lambda\cdot x)$ as well and the L^∞ norm of $P(x+x_0)$ is the same as the L^∞ norm of $P(x)$. Finally (7) is equivalent to the seemingly stronger property

$$\Big\|\sum_{\lambda\in\Lambda} \phi(\lambda)c(\lambda)\exp(2\pi i\lambda \cdot x)\Big\|_\infty \le C\Big\|\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda \cdot x)\Big\|_\infty$$

for any $P(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,\lambda\cdot x)$. This requirement obviously defines an algebra.

Another equivalent formulation of $\mathcal{M}(\Lambda)$ is given now. Let us consider the linear form $\mathcal{L}_\phi$ on $\mathcal{T}_\Lambda$ which is defined by $\mathcal{L}_\phi(P) = \sum_{\lambda\in\Lambda} c(\lambda)\phi(\lambda)$ if $P = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda \cdot x)$. With these notations (7) can be rewritten as

$$|\mathcal{L}_\phi(P)| \le C\|P\|_\infty \qquad (\forall P \in \mathcal{T}_\Lambda). \tag{8}$$

Let $\Gamma \subset \mathbb{R}^n$ be the additive group generated by Λ and let G be the dual group of Γ in the sense of Pontryagin duality. Then any trigonometric sum $P \in \mathcal{T}_\Lambda$ is the restriction to $\mathbb{R}^n$ of a continuous function, in fact a trigonometric sum F on G. Hahn–Banach's theorem and the Riesz–Markov–Kakutani representation theorem imply the following:

Lemma 2.1. *For any $\phi \in \mathcal{M}(\Lambda)$ there exists a Radon measure μ on G such that $\widehat{\mu} = \phi$ on Λ and $\|\mu\| = \|\phi\|_{\mathcal{M}(\Lambda)}$. Conversely any Radon measure μ on G defines a $\phi \in \mathcal{M}(\Lambda)$ by $\widehat{\mu} = \phi$ on Λ.*

In other words $\mathcal{M}(\Lambda)$ is the restriction algebra to Λ of the Fourier–Stieltjes transforms of the Radon measures on G. Here are some other remarkable properties of $\mathcal{M}(\Lambda)$.

Lemma 2.2. *For any continuous function ϕ on Λ we have*

$$\|\phi\|_{\mathcal{M}(\Lambda)} = \sup_{F \subset \Lambda} \|\phi\|_{\mathcal{M}(F)} \tag{9}$$

where the upper bound in the right hand side of (9) *is computed over the finite subsets F of Λ.*

Lemma 2.2 is tautological.

Lemma 2.3. *When Λ is a finite set we have $\mathcal{M}(\Lambda) = A(\Lambda) = B(\Lambda)$ isometrically.*

The only non trivial piece of Lemma 2.3 is $\|\phi\|_{A(\Lambda)} \le \|\phi\|_{\mathcal{M}(\Lambda)}$. Let us prove this remark. There exists a Radon measure μ on G such that $\|\mu\| = \|\phi\|_{\mathcal{M}(\Lambda)}$ and $\widehat{\mu} = \phi$ on Λ. But μ is the weak-star limit of a sequence P_j of trigonometric sums on G such that $\|P_j\|_1 \le \|\mu\|$. Therefore the mean value over $\mathbb{R}^n$ of $|P_j|$ does not exceed $\|\mu\|$. Finally a sequence f_j in $L^1(\mathbb{R}^n)$ is defined by $f_{j,R}(x) = R^{-n} w(x/R) P_j(x)$ where w is the indicator function of the unit cube. We have $\|f_{j,R_j}\|_1 \le \|\phi\|_{\mathcal{M}(\Lambda)} + \epsilon$ and $\widehat{f}_{j,R_j} \to \phi$ on Λ when $R = R_j$ and $j \to \infty$. Since all the norms on a finite dimensional vector space are equivalent the sequence $\widehat{f}_{j,R_j}$ converges to ϕ in $A(\Lambda)$ which implies $\|\phi\|_{A(\Lambda)} \le \|\phi\|_{\mathcal{M}(\Lambda)}$ as announced.

Corollary 2.1. *For any continuous function ϕ on Λ we have*

$$\|\phi\|_{\mathcal{M}(\Lambda)} = \sup_{F \subset \Lambda} \|\phi\|_{A(F)}. \tag{10}$$

Here are four examples where the Banach algebra $\mathcal{M}(\Lambda)$ can be easily detailed. If $\Lambda = \mathbb{Z}$ we obviously have $\mathcal{M}(\mathbb{Z}) = B(\mathbb{Z})$. On the opposite direction if the real numbers $\lambda \in \Lambda$ are linearly independent over $\mathbb{Q}$ we have $\mathcal{M}(\Lambda) = l^\infty(\Lambda)$. Therefore $\mathcal{M}(\Lambda) = B(\Lambda)$ if and only if Λ is a Sidon set [9], [16]. Let $\alpha > 0$ be an irrational number and consider $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$. Let $\Lambda_1 = \mathbb{Z}$ and $\Lambda_2 = \alpha\mathbb{Z} \setminus \{0\}$. Then we have $A(\Lambda) = A(\Lambda_1) \oplus A(\Lambda_2)$, $\mathcal{M}(\Lambda) = B(\Lambda_1) \oplus B(\Lambda_2)$, but $\mathcal{M}(\Lambda) \ne B(\Lambda)$. For instance the weak character χ on Λ which is defined by $\chi(k) = 1$, $\chi(\alpha k) = (-1)^k$, $k \in \mathbb{Z}$, belongs to $\mathcal{M}(\Lambda)$ but does not belong to $B(\Lambda)$. Weak characters on Λ are defined in Section 4. If $\Lambda = \{\sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$ then $\mathcal{M}(\Lambda)$ lies in between $B(\Lambda)$ and $l^\infty(\Lambda)$. These results will be explained by Theorem 3.1.

We return to the Bochner–Schoenberg–Eberlein property of S-E. Takahasi and O. Hatori (Definition 2.2).

Definition 2.7. *A closed set $\Lambda \subset \mathbb{R}^n$ satisfies Bochner–Schoenberg–Eberlein's property if the Banach algebra $A(\Lambda)$ satisfies Bochner–Schoenberg–Eberlein's property.*

A multiplier f of $A(\Lambda)$ is a continuous function on Λ such that for any $u \in A(\Lambda)$ the pointwise product fu still belongs to $A(\Lambda)$. If $u \in A(\Lambda)$ and $v \in B(\Lambda)$ then the pointwise product uv belongs to $A(\Lambda)$. But for some closed sets Λ there exist multipliers of $A(\Lambda)$ which do not belong to $B(\Lambda)$. Then Definition 2.2 can be rewritten as follows:

Proposition 2.3. *Let $\Lambda \subset \mathbb{R}^n$ be a closed set. The restriction algebra $A(\Lambda)$ satisfies the Bochner–Schoenberg–Eberlein property if and only if any $\phi \in \mathcal{M}(\Lambda)$ is a multiplier of $A(\Lambda)$.*

Bochner's property implies Bochner–Schoenberg–Eberlein property since $B(\Lambda)$ is contained in $\mathcal{M}(\Lambda)$. The following result illustrates this remark.

Theorem 2.3. *Any locally finite set $\Lambda \subset \mathbb{R}^n$ satisfies Bochner–Schoenberg–Eberlein property.*

In full contrast a locally fine set Λ satisfies Bochner's property if and only if Λ is a coherent set of frequencies. By Proposition 2.3 it suffices to prove that any $\phi \in \mathcal{M}(\Lambda)$ is a multiplier of $A(\Lambda)$. More precisely for any $f \in A(\Lambda)$ we shall prove that

$$\|\phi f\|_{A(\Lambda)} \leq \|f\|_{A(\Lambda)} \|\phi\|_{\mathcal{M}(\Lambda)}. \tag{11}$$

We begin with a trivial lemma:

Lemma 2.4. *If $\Lambda \subset \mathbb{R}^n$ is locally finite then the vector space $V \subset S(\Lambda)$ consisting of the finitely supported functions is dense in $A(\Lambda)$.*

Indeed let $g \in L^1(\mathbb{R}^n)$ be a non negative function with the two following properties: (a) $\int g = 1$ and (b) $\widehat{g}$ is compactly supported. Let $g_j(x) = j^n g(jx)$. Then let $f \in A(\Lambda)$. We have $f = \widehat{F}$ where $F \in L^1(\mathbb{R}^n)$. Then $F * g_j$ converges to F in $L^1(\mathbb{R}^n)$ which implies that the product $f\widehat{g}_j$ converges to f in $A(\Lambda)$. Lemma 2.4 follows since $\widehat{g}_j(x) = \widehat{g}(x/j)$. We return to (11). By density it suffices to check this estimate when u has a finite support. We then use a simple observation.

Lemma 2.5. *If $F \subset \Lambda$ is a finite set, then for any $\epsilon > 0$ there exists a finite set $T \subset \Lambda$ such that for any $g \in S(\Lambda)$ supported by F we have $\|g\|_{A(\Lambda)} \leq \|g\|_{A(T)} + \epsilon$.*

Indeed there exists a compactly supported function $\phi \in A(\mathbb{R}^n)$ such that $\phi = 1$ on F and such that $\|\phi\|_{A(\mathbb{R}^n)} \leq 1 + \epsilon$. Let T be the support of ϕ. By definition of the norm in $A(T)$ there exists a function h in $A(\mathbb{R}^n)$ such that $h = g$ on T and $\|h\|_{A(\mathbb{R}^n)} \leq \|g\|_{A(T)} + \epsilon$. Since $g = \phi h$ we have

$$\|g\|_{A(\Lambda)} \leq \|\phi\|_{A(\Lambda)} \|h\|_{A(\Lambda)} \leq (1+\epsilon)(\|g\|_{A(T)} + \epsilon) \tag{12}$$

which concludes the proof.

We return to the proof of (11). By Lemma 2.4 it suffices to prove (11) when f is compactly supported. Let F be the support of f and let T be defined by Lemma 2.5. It suffices to show that

$$\|\phi f\|_{A(T)} \leq \|f\|_{A(T)} \|\phi\|_{\mathcal{M}(T)}. \tag{13}$$

But (13) is trivial since $\mathcal{M}(T) = A(T)$ isometrically and $A(T)$ is a Banach algebra. This ends the proof of Theorem 2.3. Let us observe that conversely any multiplier of $A(\Lambda)$ belongs to $\mathcal{M}(\Lambda)$ if Λ is locally finite.

Haskell P. Rosenthal proved the following result:

Theorem 2.4. *Let $E \subset \mathbb{R}$ be a compact set. If for any $x \in E$ and for any $\epsilon > 0$ the intersection $E \cap [x-\epsilon, x+\epsilon]$ has a positive Lebesgue measure, then E satisfies Bochner's property.*

3. Coherent sets of frequencies

Coherent set of frequencies were defined and studied by Jean-Pierre Kahane in [9]. From now on $\Lambda \subset \mathbb{R}^n$ will be a closed and discrete set. In other words Λ is locally finite: for any ball B centered at 0 with radius R the intersection $\Lambda \cap B$ is a finite set of cardinality $C(R)$. Let $\mathcal{T}_\Lambda$ be the vector space consisting of all finite trigonometric sums $P(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i\, x \cdot \lambda)$ whose frequencies belong to Λ.

Definition 3.1. *A set $\Lambda \subset \mathbb{R}^n$ is a coherent set of frequencies if there exists a compact set $K \subset \mathbb{R}^n$ and a constant C such that for any $P \in \mathcal{T}_\Lambda$ one has*

$$\|P\|_\infty \le C \sup_{x \in K} |P(x)|. \tag{14}$$

This property was named $Q(\Lambda)$ in [9]. Kahane's motivation was the theory of mean periodic functions. Given a locally finite set $\Lambda \subset \mathbb{R}^n$ one denotes by $\mathcal{M}P_\Lambda$ the closure of $\mathcal{T}_\Lambda$ for the topology of uniform convergence on compact sets. Then two cases can occur. Either $\mathcal{M}P_\Lambda$ coincides with the space of all continuous functions on $\mathbb{R}^n$. If it occurs Λ will be called a "bad set". An example of a bad set in one dimension is $\Lambda = \{\sqrt{2}, \ldots, \sqrt{n}, \ldots\}$. If Λ is not a "bad set", we say that Λ is a "good set". If Λ is a "good set" any $f \in \mathcal{M}P_\Lambda$ is, by definition, a mean periodic function whose spectrum is simple and contained in Λ. By Hahn–Banach theorem Λ is a "good set" if and only if there exists a non trivial compactly supported Radon measure μ whose Fourier transform vanishes on Λ. Then any $f \in \mathcal{M}P_\Lambda$ satisfies the convolution equation $f * \mu = 0$. Let $\mathcal{A}P_\Lambda$ denote the closure of $\mathcal{T}_\Lambda$ for the topology of uniform convergence over $\mathbb{R}^n$. Then any $f \in \mathcal{A}P_\Lambda$ is an almost periodic function whose spectrum is contained in Λ. We obviously have $\mathcal{A}P_\Lambda \subset \mathcal{M}P_\Lambda$. A simple one dimensional example of a good set Λ for which $\mathcal{A}P_\Lambda \neq \mathcal{M}P_\Lambda$ is given by $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$ where $\alpha \notin \mathbb{Q}$. Then any $f \in \mathcal{A}P_\Lambda$ can be uniquely written as $f = f_0 + f_1$ where f_0 is a continuous function, periodic of period 1 and f_1 is a continuous function, periodic of period $1/\alpha$. But a mean periodic function $f \in \mathcal{M}P_\Lambda$ cannot be split, in general, into such a sum $f = f_0 + f_1$. Kahane advised us to study the locally finite sets Λ such that $\mathcal{A}P_\Lambda = \mathcal{M}P_\Lambda$. Kahane proved that this occurs if and only if Λ is a coherent set of frequencies [9].

A coherent set of frequencies Λ is uniformly discrete: there exists a positive β such that $\lambda \neq \lambda',\ \lambda, \lambda' \in \Lambda$, implies $|\lambda - \lambda'| \ge \beta$. If (14) is satisfied for a pair (K, C) it is also satisfied for any pair (L, C) when $K \subset L$. We then denote by C_j the lower bound of the constants C figuring in the right hand side of (14) when K is the ball B_j of radius $j \ge 1$ centered at 0.

Definition 3.2. *With the preceding notations $\alpha(\Lambda)$ is defined as the limit of the decreasing sequence C_j as j tends to infinity.*

We write $\alpha(\Lambda) = \infty$ if Λ is not a coherent set of frequencies. The following lemma will be used below:

Lemma 3.1. *If $\Lambda \subset \mathbb{R}^n$ is a coherent set of frequencies there exists a compact set $K \subset \mathbb{R}^n$ and a constant C such that for every bounded Radon measure μ on $\mathbb{R}^n$ there exists a Radon measure ν supported by K such that $\|\nu\| \le C\|\mu\|$ and $\widehat{\nu} = \widehat{\mu}$ on Λ.*

Lemma 3.1 is immediate and was observed in [9].

Coherent sets of frequencies have a rigid structure as the following examples show. In one dimension let $\lambda_k = k + r_k$, $-1/2 \le r_k < 1/2$, and $\Lambda = \{\, \lambda_k,\, k \in \mathbb{Z}\,\}$. If F is a finite set of real numbers and if $r_k \in F$, $k \in \mathbb{Z}$, then Λ is a coherent set of frequencies. If $r_k = \beta \sin(2\pi\alpha k)$, $0 < \beta < 1/2$, and $\alpha \notin \mathbb{Q}$ it is not the case [18]. Let $\{x\} \in [0, 1)$, be the fractional part of a real number x and let $r_k = \beta\{2\pi\alpha k\}$, $0 < \beta < 1/2$. Then Λ is a coherent set of frequencies [17], [18].

Here is our main theorem:

Theorem 3.1. *Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. Then the following two properties of Λ are equivalent:*

(a) *Λ satisfies Bochner's property.*

(b) *Λ is a coherent set of frequencies.*

Moreover $\alpha(\Lambda) = \omega(\Lambda)$.

The implication $(b) \Rightarrow (a)$ is trivial. Let us assume that a $\phi \in l^\infty(\Lambda)$ satisfies (4). The linear form $\mathcal{L}_\phi$ defined on $\mathcal{T}_\Lambda$ by $\mathcal{L}_\phi(P) = \sum_{\lambda\in\Lambda} c(\lambda)\phi(\lambda)$ when $P = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda \cdot x)$ extends by (14) and Hahn–Banach theorem to a continuous linear form on the Banach space of continuous functions on K. Then the Riesz–Markov–Kakutani representation theorem provides us with a measure μ supported by K whose Fourier transform coincides with $\tilde{\phi}$ on Λ where $\tilde{\phi}(x) = \phi(-x)$. This argument yields $\omega(\Lambda) \le \alpha(\Lambda)$. The proofs of $(a) \Rightarrow (b)$ and of $\alpha(\Lambda) \le \omega(\Lambda)$ are more involved and are given in the following sections.

4. Weak characters

Weak characters on Λ which play a key role in the proof of Theorem 3.1 are defined in this section. Let $\Lambda \subset \mathbb{R}^n$ be a locally finite set and let Γ be the additive group generated by Λ. In other terms Γ is the set of all finite sums $\sum m_j\lambda_j$ with $m_j \in \mathbb{Z}$ and $\lambda_j \in \Lambda$. The group Γ is equipped with the discrete topology (even if it is a dense subgroup of $\mathbb{R}^n$).

Definition 4.1. *Let $\mathbb{T}$ be the multiplicative group of complex numbers of modulus 1. A weak character $\chi\colon \Lambda \mapsto \mathbb{T}$ is the restriction to Λ of a homomorphism $\chi\colon \Gamma \mapsto \mathbb{T}$.*

We then have $\chi(x + y) = \chi(x)\chi(y)$, $x, y \in \Gamma$. No continuity is required on a weak character. Trivial examples of weak characters are given by "strong characters". A strong character χ is a weak character on $\mathbb{R}^n$ which is continuous on $\mathbb{R}^n$. We then have $\chi(x) = \exp(2\pi i\, x \cdot \omega)$, $\omega \in \mathbb{R}^n$. If Λ is a lattice then any weak character on Λ is a strong character. The collection of all weak characters on Λ is the compact abelian group G which was introduced in Section 2, Lemma 2.1. Since Γ is countable, G is a metrizable compact group. In the sense given by Pontryagin's duality G is the dual group of Γ. For any $\omega \in \mathbb{R}^n$ the character $x \mapsto \exp(2\pi i x \cdot \omega)$ on Γ is denoted by $h(\omega)$. Then the mapping $h : \mathbb{R}^n \mapsto G$ has two remarkable properties. First $h(\mathbb{R}^n)$ is dense in G and secondly any $P \in \mathcal{T}_\Lambda$ extends continuously from $\mathbb{R}^n$ to G. For every $\phi \in \mathcal{M}(\Lambda)$ there exists a Radon measure μ on G such that $\widehat{\mu} = \phi$ on Λ and $\|\mu\| = \|\phi\|_{\mathcal{M}(\Lambda)}$. This is

Lemma 2.1. The density of $h(\mathbb{R}^n)$ in the metrizable group G provides us with a simple construction of weak characters which is detailed in the following lemma:

Lemma 4.1. *Let $\omega_j \in \mathbb{R}^n$ be a sequence such that the limit $\lim_{j\to\infty} \exp(2\pi i\lambda \cdot \omega_j)$ exists for every $\lambda \in \Lambda$. Let us denote by $\chi(\lambda)$ this limit. Then χ is a weak character on Λ. Conversely for every weak character χ there exists a sequence $\omega_j \in \mathbb{R}^n$ such that χ is given by this representation.*

Moreover if $\omega_j \in \mathbb{R}^n$ is an arbitrary sequence there exists a subsequence ω_{j_m} and a weak character χ such that $\lim_{m\to\infty} \exp(2\pi i\lambda \cdot \omega_{j_m}) = \chi(\lambda)$.

Lemma 4.2. *Let χ be a weak character on Λ. Then $\|\chi\|_{\mathcal{M}(\Lambda)} = 1$.*

The proof of Lemma 4.2 relies on the following observation.

Lemma 4.3. *Let Λ be a locally finite set. Then if $f_j \in \mathcal{M}(\Lambda)$ is a bounded sequence and if $f_j \to f$ pointwise on Λ then $f \in \mathcal{M}(\Lambda)$ and $\|f\|_{\mathcal{M}} \le \liminf \|f_j\|_{\mathcal{M}}$.*

It suffices to prove that (7) holds true for the function f. We know that (7) holds true for f_j with a uniform constant C. Let σ be a finite linear combination of Dirac masses and let F be the finite support of σ. Then $f_j \to f$ on F and we can pass to the limit in (7). This ends the proof of Lemma 4.3.

We return to Lemma 4.2. If χ is a weak character there exists a sequence χ_j of strong characters which tend to χ pointwise on Λ. We have $\|\chi_j\|_{\mathcal{M}(\Lambda)} = 1$ and Lemma 4.3 implies $\|\chi\|_{\mathcal{M}(\Lambda)} \le 1$. But $\|\chi\|_{\mathcal{M}(\Lambda)} \ge 1$ is obvious since the norm in $\mathcal{M}(\Lambda)$ is larger than the l^∞ norm. This concludes the proof of Lemma 4.2. The converse implication in Lemma 4.2 is a characterization of weak characters which is inspired by a remarkable paper by K. de Leeuw and Y. Katznelson [4].

Theorem 4.1. *Let $\chi \in l^\infty(\Lambda)$. If $|\chi(\lambda)| = 1$, $\forall\lambda \in \Lambda$, and $\|\chi\|_{\mathcal{M}(\Lambda)} = 1$ then $\chi = c_0\chi_0$ where c_0 is a constant of modulus 1 and χ_0 is a weak character on Λ.*

The proof of Theorem 4.1 is borrowed from [4]. The problem is translation invariant and one can assume $0 \in \Lambda$. By a suitable choice of the constant c_0 we can assume $\chi(0) = 1$. By Lemma 2.1 there exists a Radon measure μ on G such that $\|\mu\| = 1$ and $\widehat{\mu}(\lambda) = \chi(\lambda)$, $\forall\lambda \in \Lambda$. Since $\int_G d\mu = \widehat{\mu}(0) = 1 = \|\mu\|$ the measure μ is a probability measure. We then use Lemma 2.1 of [4]. It tells us that for every probability measure μ on G the set $S = \{\, \gamma \in \Gamma;\ |\widehat{\mu}(\gamma)| = 1 \,\}$ is a subgroup of Γ and that $\widehat{\mu}(\lambda)$ is multiplicative on S. In our situation Λ is contained in S. Therefore $S = \Gamma$ and χ is a weak character on Λ.

Lemma 2.1 of [4] is mentioned as folklore. Here is a proof. For $\gamma \in S$ we define $\chi(\gamma)$ by $\chi(\gamma) = \int_G \exp(-2\pi i\gamma \cdot x)\, d\mu$. If $\gamma \in S$ we have $|\chi(\gamma)| = 1$. Since μ is a probability measure $|\widehat{\mu}(\gamma)| = 1$ implies $\overline{\chi}(\gamma) \exp(-2\pi i\gamma \cdot x) = 1$ for μ almost all x. Then for $\gamma, \gamma' \in S$ we have $\exp(2\pi i\gamma \cdot x) = \chi(\gamma)$ and $\exp(2\pi i\gamma' \cdot x) = \chi(\gamma')$ almost everywhere to respect with μ. It obviously implies $\chi(\gamma)\chi(\gamma') = \exp(2\pi i(\gamma + \gamma') \cdot x)$ μ almost everywhere. Integrated respect to μ this gives $\widehat{\mu}(\gamma + \gamma') = \chi(\gamma)\chi(\gamma')$. But $|\chi(\gamma)| = |\chi(\gamma')| = 1$. Therefore $\gamma + \gamma' \in S$ and $\widehat{\mu}(\gamma + \gamma') = \widehat{\mu}(\gamma)\widehat{\mu}(\gamma')$. On the other hand $\widehat{\mu}(-\gamma) = \overline{\widehat{\mu}}(\gamma)$. Therefore $\gamma \in S$ implies $-\gamma \in S$ and S is a group as announced.

A similar approach can be used in the problem of extension of positive definite functions raised in Section 2. Let $E \subset \mathbb{R}^n$ be a set and $\Lambda = E - E$. Then a positive definite function on Λ is defined by the following condition.

Definition 4.2. *A function ϕ defined on Λ is positive definite if for any N and any $x_j \in E$, $c_j \in \mathbb{C}$, $1 \leq j \leq N$, we have $\sum_1^N \sum_1^N c_j \bar{c}_k \phi(x_j - x_k) \geq 0$.*

The following lemma is seminal in our approach to the extension issue.

Lemma 4.4. *Let Γ be the subgroup of $\mathbb{R}^n$ generated by E. Then any weak character χ on Γ is positive definite on Λ.*

Indeed $\sum_1^N \sum_1^N c_j \bar{c}_k \chi(x_j - x_k) = \sum_1^N \sum_1^N c_j \bar{c}_k \chi(x_j)\overline{\chi}(x_k) = |\sum_1^N c_j \chi(x_j)|^2$.

We say that $\Lambda \subset \mathbb{R}^n$ satisfies property $\mathcal{R}$ if any continuous positive definite function F on Λ is the restriction to Λ of a continuous positive definite function G on $\mathbb{R}^n$. We have $0 \in \Lambda$. Therefore property $\mathcal{R}$ implies that any positive definite continuous function F on Λ such that $F(0) = 1$ is the restriction to Λ of a characteristic function. Mark Krein [11] proved that $[-1, 1]$ satisfies property $\mathcal{R}$. Rudin [21] proved that $[-1, 1]^n$, $n \geq 2$, does not satisfy $\mathcal{R}$. We now focus on locally finite sets Λ and the continuity assumption disappears. For instance Λ is locally finite if E is a harmonious set. This is detailed in Section 5.

Proposition 4.1. *Let E be a locally finite set such that $\Lambda = E - E$ is also a locally finite set. Then property $\mathcal{R}$ implies that Λ is contained in a lattice.*

Let χ be a weak character on Λ. Lemma 4.4 and $\mathcal{R}$ imply that χ coincides on Λ with the Fourier transform of a non negative Radon measure μ. We have $\widehat{\mu}(0) = 1$. Therefore μ is a probability measure. Katznelson's lemma implies the following property: The set H defined by $|\widehat{\mu}| = 1$ is a subgroup of $\mathbb{R}^n$ and $\widehat{\mu}$ is multiplicative on H. But $\chi = \widehat{\mu}$ on Λ implies $\chi = \widehat{\mu}$ on Γ since χ is multiplicative on Γ. Therefore χ is uniformly continuous on Γ. It implies that χ is a strong character. We proved that any weak character on Γ is a strong character. It implies that Γ is a discrete subgroup of $\mathbb{R}^n$. Therefore Λ is contained in a lattice.

Definition 4.3. *A closed and discrete set Λ satisfies the weak Bochner's property if any weak character on Λ belongs to $B(\Lambda)$. We then define $\gamma(\Lambda) = \sup \|\chi\|_{B(\Lambda)}$ where this upper bound is computed on all weak characters χ on Λ.*

If any weak character belongs to $B(\Lambda)$ it will be proved that $\gamma(\Lambda)$ is finite. We are now ready for the principal result of this note.

Theorem 4.2. *Let $\Lambda \subset \mathbb{R}^n$ a closed and discrete set. Then the following three properties are equivalent:*

- *(a) Λ satisfies the weak Bochner's property;*
- *(b) Λ satisfies Bochner's property;*
- *(c) Λ is a coherent set of frequencies.*

Moreover $\alpha(\Lambda) = \omega(\Lambda) = \gamma(\Lambda)$.

The relevance of $\gamma(\Lambda)$ is discussed before proving Theorem 4.2. We always have $\gamma(\Lambda) \geq 1$. Moreover $\gamma(\Lambda) = 1$ if Λ is a quasi-crystal (Section 6). Here is an example of a coherent set of frequencies for which $\gamma(\Lambda) > 1$.

Theorem 4.3. *Let $\theta \notin \mathbb{Q}$ and $M_\theta = \theta\mathbb{Z} \setminus (\mathbb{Z} + [-1/5, 1/5])$. Let $\Lambda^\theta = \mathbb{Z} \cup M_\theta$. Then we have $\gamma(\Lambda^\theta) > 1$.*

To prove Theorem 4.3 it suffices to construct a weak character χ on Λ^θ such that $\|\chi\|_{B(\Lambda^\theta)} > 1$. If χ_0 is a strong character we obviously have $\|\chi_0\|_{B(\Lambda^\theta)} = 1$. Let $\phi > 0$ such that $1, \theta$, and ϕ are linearly independent over $\mathbb{Q}$. Our weak character χ is defined by $\chi = 1$ on $\mathbb{Z}$ and $\chi(\theta k) = \exp(2\pi i k\phi)$ on M_θ. Let us argue by contradiction and assume that $\|\chi\|_{B(\Lambda^\theta)} = 1$. Then for every $\epsilon > 0$ there exists a Radon measure μ such that $\widehat{\mu} = \chi$ on Λ^θ and $\|\mu\| \leq 1 + \epsilon$. A contradiction will be reached if $0 < \epsilon < 10^{-2}$. The following lemma is a first step to the proof.

Lemma 4.5. *If z_k, $k \in \mathbb{Z}$, is a sequence of complex numbers such that $\sum_{k\in\mathbb{Z}} z_k = 1$ and $\sum_{k\in\mathbb{Z}} |z_k| = 1 + \epsilon$, then there exists a sequence $p_k \in [0, 1]$ such that*

(1) $\sum_{k\in\mathbb{Z}} p_k = 1$;
(2) $z_k = p_k + r_k$;
(3) $\sum_{k\in\mathbb{Z}} |r_k| \leq 3\sqrt{\epsilon}$.

To prove Lemma 4.5 we write $z_k = x_k + iy_k$ and we have $\sum_{k\in\mathbb{Z}}(|z_k| - x_k) = \epsilon$. We then observe that for any complex number $z = x+iy$ we have $\big||z|-z\big| \leq \sqrt{3|z|(|z| - x)}$. Therefore $\sum_{k\in\mathbb{Z}} \big||z_k| - z_k\big| \leq \sum_{k\in\mathbb{Z}} \sqrt{3|z_k|(|z_k| - x_k)}$. By Cauchy-Schwarz we obtain $\sum_{k\in\mathbb{Z}} \big||z_k| - z_k\big| \leq \sqrt{3\epsilon(1 + \epsilon)}$. Finally it suffices to set $p_k = (1 + \epsilon)^{-1}|z_k|$. Then $\sum p_k = 1$ and $\sum_{k\in\mathbb{Z}} |r_k| \leq 3\sqrt{\epsilon}$ are immediate.

Lemma 4.6. *If μ is a bounded Radon measure on the real line, if $\|\mu\| \leq 1 + \epsilon$ and $\widehat{\mu} = 1$ on $\mathbb{Z}$, then $\mu = \sum_{k\in\mathbb{Z}} p_k\delta_k + \rho$ where $p_k \in [0, 1]$, $\sum_{k\in\mathbb{Z}} p_k = 1$, and $\|\rho\| \leq 3\sqrt{\epsilon}$.*

Let $\nu = \sum_{k\in\mathbb{Z}} z_k\delta_k$ be the restriction of μ to $\mathbb{Z}$ and let $\rho = \mu - \nu$. Let $\tau = \sum_{k\in\mathbb{Z}} \delta_k$ be the Dirac comb. Then $\widehat{\mu} = 1$ on $\mathbb{Z}$ implies $\mu * \tau = \tau$. It yields $\nu * \tau = \tau$ and $\widehat{\nu} = 1$ on $\mathbb{Z}$ since $\rho * \tau$ cannot charge $\mathbb{Z}$. We end with $\sum_{k\in\mathbb{Z}} z_k = 1$. Obviously $\|\nu\| \leq \|\mu\| \leq 1 + \epsilon$ which implies $\sum_{k\in\mathbb{Z}} |z_k| \leq 1 + \epsilon$. It suffices now to use Lemma 4.5 which ends the proof of Lemma 4.6.

Let us return to the proof of Theorem 4.3. Let us assume that $\|\mu\| \leq 1 + \epsilon$, $\widehat{\mu} = 1$ on $\mathbb{Z}$, and

$$\widehat{\mu}(m\theta) = \exp(2\pi i m\phi) \text{ when } m \in \mathbb{Z},\ |m\theta - l| \geq 1/5,\ l \in \mathbb{Z}. \tag{15}$$

Lemma 4.6 yields $\mu = \nu + \rho$ where $\nu = \sum_{k\in\mathbb{Z}} p_k\delta_k + \rho$ and $\|\rho\| \leq 3\sqrt{\epsilon}$. We now consider the 1 periodic continuous function $F(x) = \sum_{k\in\mathbb{Z}} p_k \exp(-2\pi i k x)$. Then $\widehat{\mu}(x) = F(x) + \eta(x)$ where $\|\eta\|_\infty \leq 3\sqrt{\epsilon}$. Next (15) implies

$$F(m\theta) = \exp(2\pi i m\phi) + \epsilon_m \tag{16}$$

with $|\epsilon_m| \leq 3\sqrt{\epsilon}$ if $|m\theta - l| \geq 1/5$, $l \in \mathbb{Z}$. Since 1, θ, and ϕ are linearly independent over $\mathbb{Q}$ there exist two sequences k_j, k'_j of integers such that $k_j\theta = 1/2 + m_j + o(1)$ and $k_j\phi = -1/4 + n_j + o(1)$, $m_j, n_j \in \mathbb{Z}$ but $k'_j\theta = 1/2 + m'_j + o(1)$ and $k'_j\phi =$

$1/4 + n'_j + o(1)$, $m'_j, n'_j \in \mathbb{Z}$. Therefore these k_j and k'_j satisfy $|k\theta - m| \geq 1/5$ if j is large enough. Since F is a continuous function, (16) and $k'_j\phi = 1/4 + n'_j + o(1)$ imply $F(1/2) = i$. But $k_j\phi = -1/4 + n_j + o(1)$ and (16) imply $F(1/2) = -i$. We reach the expected contradiction when ϵ is small enough.

5. Proof of Theorem 4.2

Let us begin with some easy remarks. The proof of $(c) \Rightarrow (b)$ was already given in Section 3. The proof of $(b) \Rightarrow (a)$ is trivial since we know that any weak character satisfies $\|\chi\|_{\mathcal{M}(\Lambda)} = 1$. We obviously have $\gamma(\Lambda) \leq \omega(\Lambda)$.

We now prove $(a) \Rightarrow (c)$. We argue by contradiction. We assume that Λ is not a coherent set of frequencies and we construct a weak character on Λ which does not belong to $B(\Lambda)$. This construction relies on Theorem 5.1. In this theorem Λ is an arbitrary set. The hypothesis that Λ is locally finite will be needed to conclude the proof of Theorem 4.2.

Theorem 5.1. *Let us assume that $\Lambda \subset \mathbb{R}^n$ is not a coherent set of frequencies. Then there exist a sequence $P_j \in \mathcal{T}_\Lambda$, $j \in \mathbb{N}$, and a sequence $x_j \in \mathbb{R}^n$ such that the four following properties hold:*

(i) $\|P_j\|_\infty = 1$;
(ii) $\sup_{\{|x| \leq j\}} |P_j(x)| \leq 1/j$;
(iii) $|P_k(x_j)| \geq 1 - (2^{-k} + \cdots + 2^{-j})$ *if* $1 \leq k \leq j-1$;
(iiii) $|P_j(x_j)| \geq 1 - 2^{-j}$.

The proof of Theorem 5.1 relies on a fine lemma by N. Varopoulos. Let $\mathrm{AC}(\mathbb{R}^n)$ be the Banach algebra of all Fourier transforms of bounded atomic measures. Then $f \in \mathrm{AC}(\mathbb{R}^n)$ is an almost periodic function with an absolutely convergent Fourier series. If $\Lambda \subset \mathbb{R}^n$ is a closed set we denote by $\mathrm{AC}(\Lambda)$ the corresponding restriction algebra. We obviously have $\mathrm{AC}(\Lambda) \subset B(\Lambda)$ and $\|f\|_\infty \leq \|f\|_{B(\Lambda)} \leq \|f\|_{\mathrm{AC}(\Lambda)}$ for any $f \in \mathrm{AC}(\Lambda)$. With these notations we have:

Lemma 5.1. *Let $\Lambda \subset \mathbb{R}^n$ be a closed set and $\omega \in \mathbb{R}^n$. Let*

$$\eta(\omega) = \sup_{\lambda \in \Lambda} \big|\exp(2\pi i \omega \cdot \lambda) - 1\big|.$$

We then have

$$\big\|\exp(2\pi i \omega \cdot \lambda) - 1\big\|_{AC(\Lambda)} \leq 2\eta(\omega). \tag{17}$$

Varopoulos' lemma follows from a simple observation.

Lemma 5.2. *For any real number $\epsilon \in [0,1]$, there exists a sequence c_k, $k \in \mathbb{Z}$, of complex numbers such that*

(1) $\sum_{k \in \mathbb{Z}} |c_k| \leq 2\epsilon$;
(2) $|z| = 1$ *and* $|z - 1| \leq \epsilon$ *imply* $z - 1 = \sum_{k \in \mathbb{Z}} c_k z^k$.

Lemma 5.2 is proved in [17, Ch. IV, p. 108]. Let us return to Lemma 5.1. If $\eta(\omega) \geq 1$ (17) is trivial. We assume $\epsilon = \eta(\omega) < 1$ and we have $|\exp(2\pi i \omega \cdot \lambda) - 1| \leq \epsilon$ for any $\lambda \in \Lambda$. Then Lemma 5.2 yields $\exp(2\pi i \omega \cdot \lambda) - 1 = \sum_{k \in \mathbb{Z}} c_k \exp(2\pi i k \omega \cdot \lambda)$, $\forall \lambda \in \Lambda$, where $\sum_{k\in\mathbb{Z}}|c_k| \le 2\epsilon$. We denote by σ the atomic measure which is supported by $\omega\mathbb{Z}$ and given by $\sigma = \sum_{k\in\mathbb{Z}} c_k\delta_{-k\omega}$. We then have $\widehat{\sigma}(\lambda) = \sum_{k\in\mathbb{Z}} c_k \exp(2\pi ik\omega\cdot\lambda) = \exp(2\pi i\omega\cdot\lambda) - 1$, $\forall\lambda\in\Lambda$, and $\|\sigma\| \le 2\epsilon$ as announced.

Lemma 5.1 implies an interesting improvement on Bernstein's theorem on band limited functions. Bernstein's theorem is the following statement. If $f \in L^\infty(\mathbb{R})$ and if the Fourier transform of f is supported by $[-T,T]$ then we have $\|\frac{d}{dx}f\|_\infty \le T\|f\|_\infty$. Keeping the notations of Lemma 5.1 we have:

Theorem 5.2. *For any closed set Λ, any $f \in L^\infty(\mathbb{R}^n)$ whose Fourier transform is supported by Λ, and any $y\in\mathbb{R}^n$ we have*

$$\sup_x |f(x+y) - f(x)| \le 2\eta(y)\|f\|_\infty. \tag{18}$$

Indeed for every y, Varopoulos' lemma implies the existence of an atomic measure σ_y such that $\widehat{\sigma}_y(\lambda) = \exp(2\pi iy\cdot\lambda) - 1$, $\forall\lambda\in\Lambda$, and $\|\sigma_y\| \le 2\eta(y)$. Then $(f*\sigma_y)(x) = f(x+y) - f(x)$ which implies (18). If $\Lambda = [-T,T]$ we recover the standard Bernstein's theorem on bandlimited function.

Here is a second classical result.

Lemma 5.3. *Let $S\subset\mathbb{R}^n$ be a finite set. Then for any $\epsilon>0$ the set $M(S,\epsilon)$ defined by*

$$M(S,\epsilon) = \{\, y\in\mathbb{R}^n |\ \sup_{x\in S}|\exp(2\pi iy\cdot x) - 1| \le \epsilon\,\} \tag{19}$$

is relatively dense in $\mathbb{R}^n$.

There exists a $R = R(S,\epsilon) > 0$ such that any ball with radius R contains at least a point in $M(S,\epsilon)$.

We are now ready to prove Theorem 5.1 by induction on j. If $j=1$ we denote by P_1 any $P\in\mathcal{T}_\Lambda$ normalized by $\|P\|_\infty = 1$. The existence of x_1 such that $|P_1(x_1)| \ge 1/2$ is then obvious. We now assume that $P_1,\dots,P_j$ have been constructed as well as x_j and we construct P_{j+1} and x_{j+1}. We denote by S_j the union of the spectra of $P_1,\dots,P_j$ and we apply Theorem 5.2, Lemmas 5.1 and 5.3 with $\epsilon = 2^{-j-2}$. We end with a relatively dense set M_j such that for any $\tau\in M_j$ and $1\le k\le j$ we have

$$\|P_k(x+\tau) - P_k(x)\|_\infty \le 2^{-j-1}. \tag{20}$$

Since M_j is a relatively dense set there exists $R_j > 0$ such that for any $y\in\mathbb{R}^n$ the ball of radius R_j centered at y contains at least a point $x\in M_j$. Let $T_{j+1} = R_j + j + 1$. Let $Q_{j+1}\in\mathcal{T}_\Lambda$ such that $\|Q_{j+1}\|_\infty = 1$ and

$$\sup_{|x|\le T_{j+1}} |Q_{j+1}(x)| \le 1/(j+1). \tag{21}$$

Such a Q_{j+1} exists if Λ is not a coherent set of frequencies. Next y_{j+1} is defined by $|Q_{j+1}(y_{j+1})| \ge 1 - 2^{-j-1}$. Then there exists a x_{j+1} such that $x_{j+1}\in M_j + x_j$ and $|y_{j+1} - x_{j+1}| \le R_j$. Finally we set $P_{j+1}(x) = Q_{j+1}(x - x_{j+1} + y_{j+1})$.

It remains to prove that P_{j+1} and x_{j+1} satisfy the requirements $(i), (ii)$, and $(iiii)$ and that $P_1,\dots,P_{j+1}$ and x_{j+1} satisfy (iii). First (i) is obvious. Then (ii) follows from

$|y_{j+1} - x_{j+1}| \le R_j$ and (21). The proof of (iii) is given now. We have $\tau_j = x_{j+1} - x_j \in M_j$ and (20) yields

$$|P_k(x_{j+1}) - P_k(x_j)| \le 2^{-j-1}, \ \ 1 \le k \le j. \tag{22}$$

If $1 \le k \le j-1$ (22) and (iii) imply the required lower bound. If $k = j$ one uses (22) again and $(iiii)$. Finally $P_{j+1}(x_{j+1}) = Q_{j+1}(y_{j+1})$. Then $|Q_{j+1}(y_{j+1})| \ge 1 - 2^{-j-1}$ implies $(iiii)$. The proof of Theorem 5.1 is completed.

Since Λ is countable one can extract a subsequence x_{j_m} from the sequence x_j given by Theorem 5.1 such that $\exp(2\pi i x_{j_m} \cdot \lambda) \to \chi(\lambda)$, $\lambda \in \Lambda$, where χ is a weak character on Λ. Keeping k fixed and passing to the limit $(m \to \infty)$ in (iii) one obtains $|P_k(\chi)| \ge 1 - 2^{-k+1}$. Let us assume that there exists a Radon measure μ with a finite total mass such that $\chi = \widehat{\mu}$ on Λ. The following identity paves the road to a contradiction:

Lemma 5.4. *We have* $P_k(\chi) = \int P_k(-x)d\mu(x)$.

Indeed $P_k(x) = \sum_{\lambda\in\Lambda} c_k(\lambda)\exp(2\pi i\, x\cdot\lambda)$ and

$$\int P_k(-x)d\mu(x) = \sum_{\lambda\in\Lambda} c_k(\lambda)\widehat{\mu}(\lambda) = \sum_{\lambda\in\Lambda} c_k(\lambda)\chi(\lambda) = P_k(\chi)$$

which ends the proof of Lemma 5.4. But $\int P_k(-x)d\mu(x)$ tends to 0 by (i), (ii), and by Lebesgue's dominated convergence theorem. Since we have $|P_k(\chi)| \ge 1 - 2^{-k+1}$ we reach a contradiction. The bounded measure μ does not exist and Λ does not satisfy the weak Bochner's property. The proof of the equivalence between (a), (b), and (c) in Theorem 4.2 is completed.

The proof of $\alpha(\Lambda) = \omega(\Lambda) = \gamma(\Lambda)$ is similar. As above the non trivial piece of the proof is $\alpha(\Lambda) \le \gamma(\Lambda)$. It suffices to show that $\alpha(\Lambda) \ge 1/\epsilon$ implies $\gamma(\Lambda) \ge 1/\epsilon$ when $\epsilon \in (0,1)$. If $\alpha(\Lambda) \ge 1/\epsilon$, Λ is a "bad" coherent set of frequencies. The following lemma will end the proof of Theorem 4.2.

Lemma 5.5. *If* $\alpha(\Lambda) \ge 1/\epsilon$ *there exists a weak character* χ *on* Λ *such that, for every measure* μ *such that* $\widehat{\mu}(\lambda) = \chi(\lambda)$, $\forall\lambda \in \Lambda$, *we have* $\|\mu\| \ge 1/\epsilon$. *Therefore* $\gamma(\Lambda) \ge 1/\epsilon$.

The construction of χ follows the scheme we used for the first part of Theorem 4.2. We know that for every compact set K there exists a trigonometric sum $P \in \mathcal{T}_\Lambda$ such that $\|P\|_\infty = 1$ and $\sup_K |P(x)| \le \epsilon$. Then we have

Theorem 5.3. *There exists a sequence* $P_j \in \mathcal{T}_\Lambda$, $j \in \mathbb{N}$, *and a sequence* $x_j \in \mathbb{R}^n$ *such that the four following properties hold:*

(i) $\|P_j\|_\infty = 1$;
(ii) $\sup_{\{|x|\le j\}} |P_j(x)| \le \epsilon$;
(iii) $|P_k(x_j)| \ge 1 - (2^{-k} + \cdots + 2^{-j})$ *if* $1 \le k \le j-1$;
$(iiii)$ $|P_j(x_j)| \ge 1 - 2^{-j}$.

The proof is almost identical to the argument used for proving Theorem 5.1. We replace (21) by

$$\sup_{|x|\le T_{j+1}} |Q_{j+1}(x)| \le \epsilon.$$

The proof ends with the following lines. As above there exists a weak character χ such that $|P_k(\chi)| \geq 1 - 2^{-k+1}$. If $\widehat{\mu} = \chi$ on Λ then we have $P_k(\chi) = \int P_k(-x)d\mu(x) = I_k$. But (i) and (ii) imply $|I_k| \leq \epsilon\|\mu\| + o(1)$ as $k \to \infty$. Since $P_k(\chi) \to 1$ we have $\|\mu\| \geq 1/\epsilon$ as announced.

6. Harmonious sets, Model sets, and the Pisot set

Definition 6.1. *A locally finite set $\Lambda \subset \mathbb{R}^n$ is harmonious if any weak character χ on Λ is the uniform limit on Λ of a sequence $\chi_j(x) = \exp(2\pi i\omega_j \cdot x)$ of strong characters.*

A lattice $\Gamma \subset \mathbb{R}^n$ is harmonious since every weak character on Γ is the restriction to Γ of a strong character. Conversely if every weak character on a locally finite set Λ is the restriction to Λ of a strong character, then Λ is contained in a lattice Γ. A harmonious set is uniformly discrete. If Λ is harmonious so are $\Lambda \pm \Lambda$. This is obvious from the definition. Indeed if χ is a weak character on $\Lambda \pm \Lambda$ its restriction to Λ is a weak character on Λ. Therefore χ is a uniform limit on Λ of a sequence $\chi_j(x) = \exp(2\pi i\omega_j \cdot x)$ of strong characters. We have $\chi(x + y) = \chi(x)\chi(y)$ and it implies that χ is a uniform limit on $\Lambda + \Lambda$ of the same sequence χ_j. The same observation applies to $\Lambda - \Lambda$.

We now return to Definition 6.1. By Varopoulos' lemma (Lemma 5.1) the sequence χ_j is a Cauchy sequence in $\mathrm{AC}(\Lambda)$. Therefore χ_j converges to an element $\chi' \in \mathrm{AC}(\Lambda)$. Since χ_j converge uniformly to χ we have $\chi = \chi'$. Therefore χ belongs to $\mathrm{AC}(\Lambda) \subset B(\Lambda)$ and Λ is a coherent set of frequencies by Theorem 4.2. This is the most natural proof of the fact that harmonious sets are coherent sets of frequencies and it exemplifies the seminal role played by weak characters in this note. This argument yields the following conclusion:

Theorem 6.1. *If Λ is a harmonious set we have $\gamma(\Lambda) = 1$. It implies that for any $f \in B(\Lambda)$ we have $\|f\|_{B(\Lambda)} = \|f\|_{\mathcal{M}(\Lambda)}$.*

Indeed with the preceding notations we have $\|\chi_j\|_{B(\Lambda)} = 1$ since χ_j is a strong character. But $\|\chi - \chi_j\|_{B(\Lambda)} \to 0$ as $j \to \infty$. Therefore $\|\chi\|_{B(\Lambda)} = 1$. It implies $\|f\|_{B(\Lambda)} = \|f\|_{\mathcal{M}(\Lambda)}$ for any $f \in B(\Lambda)$.

Does there exist a coherent set of frequencies Λ which satisfies $\gamma(\Lambda) = 1$ and is not harmonious? The simplest example of a coherent set of frequencies which is not harmonious is given by $\Lambda = \{1, \theta, \theta^2, \ldots\}$ when θ is neither a Pisot number nor a Salem number. Another example is given by the set Λ^θ of Theorem 4.3. We have $\gamma(\Lambda^\theta) > 1$ which implies that Λ^θ is not harmonious.

Theorem 6.2 slightly improves on Theorem 6.1.

Theorem 6.2. *Let Λ be a harmonious set. Then for any weak character χ on Λ and for any positive ϵ there exists an atomic measure σ_ϵ on $\mathbb{R}^n$ such that $\chi = \widehat{\sigma}_\epsilon$ on Λ and $\|\sigma_\epsilon\| \leq 1 + \epsilon$. Therefore χ is the restriction to Λ of an almost periodic function on $\mathbb{R}^n$ with an absolutely convergent Fourier series.*

We cannot replace ϵ by 0 in Theorem 6.2 unless Λ is contained in a lattice. Theorem 6.2 suggests that harmonious sets are close to lattices. Theorem 6.2 is naturally related to some results by S. Hartman, C. Ryll-Nardzewski, and E. Strzelecki. They defined *interpolation sets for almost-periodic functions* in [6], [24]. A locally finite set $\Lambda \subset \mathbb{R}^n$ is an interpolation set for almost-periodic functions if any $c(\lambda) \in l^\infty(\Lambda)$ is the restriction to Λ of an almost-periodic function f. If it is the case one can impose to f to have an absolutely convergent Fourier series. This was proved by Jean-François Méla [16]. E. Strzelecki proved that an increasing sequence λ_j of real numbers satisfying $\lambda_{j+1}/\lambda_j \geq q > 1$ is an interpolation set [24]. These remarks imply that the converse of the first statement of Theorem 6.2 is wrong. Here is an example. Let θ be a transcendental number. Then set $\Lambda = \{\theta^j,\, j \geq 1\}$ is not harmonious. However any $c(\lambda) \in l^\infty(\Lambda)$ is the restriction to Λ of an almost periodic function with an absolutely convergent Fourier series. We do not know whether or not the second statement in Theorem 6.2 characterizes harmonious sets.

How does one construct harmonious sets? The "cut and projection" scheme is a partial answer [17]. Here is the recipe. A lattice $\Gamma \subset \mathbb{R}^N$ is a discrete subgroup such that the quotient group $\mathbb{R}^N/\Gamma$ is compact. Equivalently $\Gamma = A(\mathbb{Z}^N)$ where A is an invertible $N \times N$ matrix. Let $m \geq 1$, $N = n + m$, $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$. Let $\Gamma \subset \mathbb{R}^N$ be a lattice. For $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, one sets $x = p_1(X)$ and $y = p_2(X)$. We now assume that $p_1 \colon \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_2(\Gamma)$ is dense in $\mathbb{R}^m$. Recall that a compact set $K \subset \mathbb{R}^m$ is Riemann integrable if its boundary has a zero Lebesgue measure. We are now ready to define "model sets."

Definition 6.2. *Let $K \subset \mathbb{R}^m$ be a Riemann integrable compact set with a non empty interior. Then the model set $\Lambda = \Lambda(\Gamma, K)$ defined by Γ and K is*

$$\Lambda = \{\, \lambda = p_1(\gamma);\ \gamma \in \Gamma,\ p_2(\gamma) \in K \,\}. \tag{23}$$

To avoid inconsistencies in Theorem 6.3 the class of model sets is enlarged. From now on a set $\Lambda \subset \mathbb{R}^n$ is called a model set if either Λ is a lattice or if Λ is a model set of the type $\Lambda(\Gamma, K)$. As it was repeatedly mentioned a set $\Lambda \subset \mathbb{R}^n$ is relatively dense if there exists a $R > 0$ such that any ball with radius R contains at least a point belonging to Λ. Equivalently Λ is relatively dense if there exists a compact ball B such that $B + \Lambda = \mathbb{R}^n$. We have

Theorem 6.3. *A model set is harmonious. Conversely a relatively dense harmonious set Λ is contained in a sum $\Lambda_0 + F$ where Λ_0 is a model set and F is finite.*

This is proved in [17].

The distributional Fourier transform of a Dirac comb is a Dirac comb. This is the standard Poisson formula. Model sets provide examples of generalized Poisson formulas, as indicated in the following theorem [17].

Theorem 6.4. *Let Λ be a model set and ϕ be a $\mathcal{C}^\infty$ function on $\mathbb{R}^m$ which vanishes outside K. Then the measure*

$$\mu = \sum_{\gamma \in \Gamma} \phi\big(p_2(\gamma)\big)\delta_{p_1(\gamma)}$$

is supported by the model set $\Lambda(\Gamma, K)$ *and its distributional Fourier transform is the atomic measure*

$$\widehat{\mu} = c_\Gamma \sum_{\gamma^* \in \Gamma^*} \widehat{\phi}\big(-p_2(\gamma^*)\big)\delta_{p_1(\gamma^*)}$$

where Γ^* *is the dual lattice of* Γ*.*

What is missing here is the fact that this distributional Fourier transform be supported by the "dual model set". Indeed $\widehat{\mu}$ is never supported by a model set since we cannot simultaneously impose that ϕ and its Fourier transform be compactly supported. The construction of atomic measures σ which, together with $\widehat{\sigma}$ are supported by locally finite sets is not a straightforward consequence of Theorem 6.4. However Nir Lev and Alexander Olevskii achieved this construction [14] using Theorem 6.4 as an auxiliary lemma.

Michel Duneau, Denis Gratias, André Katz, and Robert Moody discovered that the quasi-crystals elaborated by Dan Shechtman are model sets [19]. A set $\Lambda \subset \mathbb{R}^n$ is called a Delone set if it is uniformly discrete and relatively dense. Jeffrey C. Lagarias proved the following theorem [12], [13].

Theorem 6.5. *Let* $\Lambda \subset \mathbb{R}^n$ *be a Delone set such that* $\Lambda - \Lambda$ *is also a Delone set. Then* Λ *is harmonious.*

Let $\theta \geq 2$ be a real number, let Λ_θ^m, $m \geq 0$, be the set of all finite sum $\sum_0^{m-1} \epsilon_k \theta^k$, $\epsilon_k \in \{0, 1\}$, and let $\Lambda_\theta = \bigcup_{m \geq 0} \Lambda_\theta^m$. Then Λ_θ is uniformly discrete and will be named the Pisot set.

Theorem 6.6. *Let us assume that* θ *is not a Pisot–Thue–Vijayaraghavan number. Then* Λ_θ *does not satisfy the weak Bochner's property.*

The proof is immediate. We know that Λ_θ is not a coherent set of frequencies (this is a trivial statement [17]) and Theorem 3.1 ends the proof.

In the opposite direction we have:

Theorem 6.7. *Let us assume that* θ *is a Pisot–Thue–Vijayaraghavan number. Then* $\omega(\Lambda_\theta) = 1$*.*

Indeed Λ_θ is then a harmonious set and Theorem 6.2 ends the proof.

7. Isomorphisms between restriction algebras

The knowledge of the restriction algebra $B(\Lambda)$ suffices to decide if Λ is a coherent set of frequencies. This is not true for $A(\Lambda)$.

Theorem 7.1. *Let* Λ_1 *and* Λ_2 *be two locally finite sets. If the Banach algebras* $B(\Lambda_1)$ *and* $B(\Lambda_2)$ *are isomorphic and if* Λ_1 *is a coherent set of frequencies, so is* Λ_2*.*

Before proving this result let us observe that it would not hold if $B(\Lambda_1)$ was replaced by $A(\Lambda_1)$ and $B(\Lambda_2)$ by $A(\Lambda_2)$. Here is a one dimensional counterexample. If Λ_1 and Λ_2 are both infinite, if the elements $\lambda \in \Lambda_1$ are linearly independent over $\mathbb{Q}$ and if the same is true for Λ_2 then $A(\Lambda_1) = c_0(\Lambda_1) = c_0(\Lambda_2) = A(\Lambda_2)$. In one dimension let Λ_1

be the set of the square roots of the prime numbers while $\Lambda_2 = \{\,\theta^j,\ j \in \mathbb{N}\,\}$ where θ is a transcendental number. Then Λ_1 is not a coherent set of frequencies while Λ_2 is a coherent set of frequencies.

We now prove Theorem 7.1. Let $J : B(\Lambda_1) \mapsto B(\Lambda_2)$ an isomorphism between these two Banach algebras. We have $J(uv) = J(u)J(v)$ for $u, v \in B(\Lambda_1)$.

Lemma 7.1. *There exists a bijection $h\colon \Lambda_2 \mapsto \Lambda_1$ such that*

$$J(u) = u \circ h \quad \text{for} \quad u \in B(\Lambda_1). \tag{24}$$

We first prove (24) when u is an idempotent. An idempotent $g \in B(\Lambda)$ satisfies $g^2 = g$ which is equivalent to $g(\lambda) \in \{0, 1\}$ for $\lambda \in \Lambda$. Since J is an algebraic isomorphism it maps an idempotent $g \in B(\Lambda_1)$ to an idempotent $J(g) \in B(\Lambda_2)$. Among such idempotents are the minimal ones which cannot be decomposed as a non trivial sum of two idempotents. These minimal idempotents are the indicator function χ_λ of a singleton $\lambda \in \Lambda_1$ or Λ_2. Finally for any $\lambda_1 \in \Lambda_1$ J maps the indicator function χ_{λ_1} of $\lambda_1 \in \Lambda_1$ to the indicator function χ_{λ_2} of $\lambda_2 \in \Lambda_2$. If $\lambda_1 \neq \lambda_1'$ then the product $\chi_{\lambda_1}\chi_{\lambda_1'} = 0$. Therefore $\chi_{\lambda_2}\chi_{\lambda_2'} = 0$ and $\lambda_2 \neq \lambda_2'$. Finally there exists a bijection $h\colon \Lambda_2 \mapsto \Lambda_1$ such that $J(f) = f \circ h$ when f is the indicator function of a singleton. We now prove (24) in full generality. If $u \in \Lambda_1$ we have for any $\lambda_1 \in \Lambda_1$

$$\chi_{\lambda_1} u = u(\lambda_1)\chi_{\lambda_1}. \tag{25}$$

We apply J to both sides of (25) which yields

$$\chi_{\lambda_2} J(u) = u(\lambda_1)\chi_{\lambda_2}. \tag{26}$$

Therefore $J(u)(\lambda_2) = u(\lambda_1)$ when $h(\lambda_2) = \lambda_1$ which ends the proof of Lemma 7.2.

Lemma 7.2. *The mapping $u \mapsto u \circ h$ is an isomorphism between $A(\Lambda_1)$ and $A(\Lambda_2)$.*

Indeed if Λ is a closed set for any $f \in A(\Lambda)$ the norm of f in $A(\Lambda)$ coincides with its norm in $B(\Lambda)$. If u has a finite support $v = u \circ h$ is finitely supported and we have $\|u \circ h\|_{A(\Lambda_2)} \simeq \|u\|_{A(\Lambda_1)}$. This extends by continuity to $A(\Lambda_1)$. It implies the following lemma:

Lemma 7.3. *With the preceding notations the mapping $f \mapsto f \circ h$ is an isomorphism between $\mathcal{M}(\Lambda_1)$ and $\mathcal{M}(\Lambda_2)$.*

Indeed any $f \in \mathcal{M}(\Lambda_2)$ is a multiplier of $A(\Lambda_2)$. Therefore for any $v \in A(\Lambda_2)$ we have $u = fv \in A\Lambda_2$. But we know that $v \circ h$ and $u \circ h$ belong to $A(\Lambda_1)$. Therefore $f \circ h$ is a multiplier of $A(\Lambda_1)$. Lemma 7.3 is proved.

Finally if Λ_1 is a coherent set of frequencies we have $\mathcal{M}(\Lambda_1) = B(\Lambda_1)$. It implies $\mathcal{M}(\Lambda_2) = B(\Lambda_2)$ by Lemmas 7.1 and 7.3. Therefore Λ_2 is a coherent set of frequencies by Theorem 3.1. It ends the proof.

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Centre Borelli, ENS-Cachan, CNRS, Université Paris-Saclay, France
Email address: yves.meyer305@orange.fr

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