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GLOBAL AND LOCAL ESTIMATES ON TRIGONOMETRIC SUMS

YVES MEYER

Abstract. Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set and let $\mathcal{C}_\Lambda$ be the space of all mean periodic functions whose spectrum is simple and contained in Λ. We estimate the behavior at infinity of these mean periodic functions $f \in \mathcal{C}_\Lambda$. The tools which are needed to solve this problem will also be used to fill a gap in a preceding paper.

1. Some problems on trigonometric sums

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set.

Definition 1.1. *The vector space of all trigonometric sums*

$$P(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x) \tag{1.1}$$

whose spectrum is contained in Λ *is denoted by* $\mathcal{T}_\Lambda$*.*

Given such a set Λ does there exist a domain of *stable uniqueness* for $\mathcal{T}_\Lambda$? A domain of *stable uniqueness* is a compact set K enjoying the following property: For every $P \in \mathcal{T}_\Lambda$ it suffices to estimate P on K to obtain a global estimate for P. If Λ is a lattice this is obviously true since every $P \in \mathcal{T}_\Lambda$ is a periodic function. If Γ is the dual lattice, every $\gamma \in \Gamma$ is a period of P. If K is a fundamental domain of Γ, P is uniquely determined by its restriction to K. Can this property be extended to some closed and discrete sets which are not lattices? This is the core of this essay.

When Λ is not a lattice our problem needs to be given a more precise formulation. An estimate is given by a functional norm. Surprisingly the answer depends on the choice of this functional norm. We first consider the case when the local estimate is given by $\sup_{x\in K}|P(x)|$ where $K \subset \mathbb{R}^n$ is a compact set. This choice of the L^∞ norm raises an

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issue which is equivalent to the problem mentioned in the abstract (Theorem 4.9). A closed and discrete set Λ is uniformly discrete if $\inf_{\lambda,\lambda'\in\Lambda,\,\lambda\neq\lambda'}|\lambda-\lambda'|>0$.

Definition 1.2. *A coherent set of frequency is a uniformly discrete set $\Lambda\subset\mathbb{R}^n$ for which there exist a compact set $K\subset\mathbb{R}^n$ and a constant C such that for every $P\in\mathcal{T}_\Lambda$ one has*

$$\sup_{x\in\mathbb{R}^n}|P(x)|\leq C\sup_{x\in K}|P(x)|. \tag{1.2}$$

Our second problem is a weighted version of (1.2). A *mild set* Λ is defined as follows:

Definition 1.3. *A closed and discrete set Λ is a mild set if there exist a continuous weight $\omega(x)\geq 1$ defined on $\mathbb{R}^n$ and a compact set K such that one has for every every $P\in\mathcal{T}_\Lambda$ and every $x\in\mathbb{R}^n$*

$$|P(x)|\leq C\omega(x)\sup_{y\in K}|P(y)|. \tag{1.3}$$

A more precise definition of mild sets will be given below (Definition 4.8). As it was observed by Kahane in [3] these two problems can be given an elegant formulation if the theory of mean periodic functions is used. This will be done in Section 4 (Theorem 4.9). A new solution to these two problems will be given in this essay. A spectral analysis of Λ (Definition 5.18) plays a key role in this solution (Theorem 9.3).

These problems can be addressed with other functional norms. Let E be a function space. If $K\subset\mathbb{R}^n$ is a compact set, the space of restrictions to K of functions in E equipped with the quotient norm will be denoted by $E(K)$. It is assumed that these restrictions are well defined. The norm in $E(K)$ is denoted by $\|\cdot\|_{E(K)}$. Our third problem is the following question. Given a closed and discrete set Λ do there exist a compact set K and a constant C such that for every trigonometric sum $P\in\mathcal{T}_\Lambda$ one has

$$\forall y\in\mathbb{R}^n,\quad \|P\|_{E(K+y)}\leq C\|P\|_{E(K)}\,? \tag{1.4}$$

The case when $E(K)=L^2(K)$ and when Λ is a simple quasi-crystal is discussed in [7], [8], and [9]. Sigrid Grepstad and Nir Lev [2] discovered that fundamental domains exist in this context. Our most ambitious problem is the following question: Do there exist a weight $\omega(x)\geq 1$ and a compact set K such that for every trigonometric sum $P\in\mathcal{T}_\Lambda$ one has

$$\forall y\in\mathbb{R}^n,\quad \|P\|_{E(K+y)}\leq \omega(y)\|P\|_{E(K)}\,? \tag{1.5}$$

2. Mean periodic functions

The Fourier transform $\mathcal{F}(f)=\widehat{f}$ of a function $f\in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n}\exp(-2\pi i x\cdot\xi)\,dx. \tag{2.1}$$

Let $\mathcal{C}(\mathbb{R}^n)$ denote the vector space of all continuous functions on $\mathbb{R}^n$, equipped with the topology of *uniform convergence on compact sets*. If f_j, $j\in\mathbb{N}^n$, is a sequence of functions in $\mathcal{C}(\mathbb{R}^n)$, this sequence converges to a function $g\in\mathcal{C}(\mathbb{R}^n)$ if and only if for every compact set K we have $\sup_{x\in K}|f_j(x)-g(x)|\to 0$, $j\to\infty$. Similarly the space $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ of locally square integrable functions on $\mathbb{R}^n$ is equipped with the topology of convergence in L^2 on compact sets. The dual space of $\mathcal{C}(\mathbb{R}^n)$ consists of all compactly supported Radon measures on $\mathbb{R}^n$. The definition of mean periodic functions was given by Laurent Schwartz in [11].

Definition 2.1. *Let $f \in \mathcal{C}(\mathbb{R}^n)$ and let $W_f \subset \mathcal{C}(\mathbb{R}^n)$ denote the closed linear span of all translates $f(\cdot - y), y \in \mathbb{R}^n$, of f. Then f is a mean periodic function if $W_f \neq \mathcal{C}(\mathbb{R}^n)$.*

This is certainly the case when W_f is finite dimensional. This happens if and only if $f(x) = P(x)\exp(2\pi i\zeta \cdot x)$ where $\zeta \in \mathbb{C}^n$ and P is a polynomial. We then say that f is an exponential polynomial. Returning to the general case, Hahn–Banach theorem implies the following:

Lemma 2.2. *A continuous function f is mean periodic if and only if there exists a non trivial compactly supported Radon measure μ such that $\mu * f = 0$.*

'Non trivial' means that μ is not the trivial zero measure. Lemma 2.2 implies that the collection of mean periodic functions is a vector space: if $f_1 * \mu_1 = 0$ and $f_2 * \mu_2 = 0$ then $\mu_1 * \mu_2 = \mu_3$ is non trivial and $(f_1 + f_2) * \mu_3 = 0$.

An exponential polynomial f satisfies $f * \mu = 0$ where μ is a weighted sum of Dirac masses. Laurent Schwartz in the one dimensional case [11] and Bernard Malgrange in the general case [6] proved the following theorem:

Theorem 2.3. *Let f be a mean periodic function. Then f is a limit in $\mathcal{C}(\mathbb{R}^n)$ of a sequence of finite sums of exponential polynomials belonging to W_f.*

Definition 2.4. *Let f be a mean periodic function. Then the spectrum $\Lambda \subset \mathbb{C}^n$ of f is the set of all $\zeta \in \mathbb{C}^n$ such that $\exp(2\pi i\zeta \cdot x) \in W_f$. The multiplicity $m(\zeta)$ of ζ is the largest degree of polynomials P such that $P(x)\exp(2\pi i\zeta \cdot x) \in W_f$. We say that the spectrum Λ is simple if $m(\zeta) = 0$ for all $\zeta \in \Lambda$.*

The spectrum of a mean periodic function is a closed set [6]. Here are some one dimensional examples of mean periodic functions. Every continuous periodic function f is mean periodic. It satisfies $f * (\delta_0 - \delta_T) = 0$ where T is the period and δ_a is the Dirac measure at a. The function $f(x) = \exp x$ is mean periodic while $f(x) = \frac{\sin x}{x}$ is not mean periodic. Any global smooth solution of a partial differential equation with constant coefficients is mean periodic. For instance the function $\psi(x) = \frac{\sin|x|}{|x|}$, $x \in \mathbb{R}^3$, is a mean periodic function since it satisfies Helmholtz equation $\Delta\psi + \psi = 0$. This example shows that there exist bounded mean periodic functions which are not almost periodic functions.

3. Almost periodic functions

The reader who is familiar with the theory of almost periodic functions is invited to skip this section and to jump to Theorem 3.8 which plays a role in the proof of Theorem 5.16.

Definition 3.1. *A continuous function $f\colon \mathbb{R}^n \to \mathbb{C}$ is almost periodic (in the sense given by Harald Bohr) if it is bounded and if for every positive ε there exists a relatively dense set Λ_ε of ε-almost periods τ for f.*

These two concepts (relatively dense and ε-almost period) are now defined. A subset $\Lambda \subset \mathbb{R}^n$ is *relatively dense* if there exists a positive R such that for every $x \in \mathbb{R}^n$ the ball $B(x, R)$ centered at x with radius R contains at least a point $\lambda \in \Lambda$.

Let f be continuous and bounded on $\mathbb{R}^n$. The L^∞ norm of f is defined by

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|. \tag{3.1}$$

Let $\varepsilon \in [0, 2)$. We say that $\tau \in \mathbb{R}^n$ is an *ε-almost period* of f if

$$\|f(\,\cdot + \tau) - f(\,\cdot\,)\|_\infty \;\leq\; \varepsilon \|f\|_\infty. \tag{3.2}$$

When equipped with the norm $\|f\|_\infty$ the space of almost periodic functions on $\mathbb{R}^n$ is a Banach space which will be denoted by $\mathcal{B}$.

Let $S \subset \mathbb{R}^n$ be an arbitrary finite set. Then the trigonometric sum

$$P(x) = \sum_{\lambda \in S} c(\lambda) \exp(2\pi i \lambda \cdot x) \tag{3.3}$$

is an almost periodic function. H. Bohr proved the following theorem:

Theorem 3.2. *Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be an almost periodic function. Then for every $\varepsilon > 0$ there exist a finite set $S(\varepsilon) \subset \mathbb{R}^n$ and a trigonometric sum*

$$P_\varepsilon(x) = \sum_{\lambda \in S(\varepsilon)} c(\lambda, \varepsilon) \exp(2\pi i \lambda \cdot x)$$

such that $\|f - P_\varepsilon\|_\infty \leq \varepsilon$.

Can we say more? Theorem 3.7 will provide us with a more precise version of Theorem 3.2. Let f be an almost periodic function. Let c_n be the inverse of the volume of the unit ball. Then the mean value of f is defined by

$$\mathcal{M}(f) = \lim_{R \to \infty} c_n R^{-n} \int_{B(x,R)} f(y)\, dy \tag{3.4}$$

and this limit is attained uniformly in x. Moreover for each $\omega \in \mathbb{R}^n$ the product $\exp(2\pi i \omega \cdot x)\, f(x)$ is also an almost periodic function. This leads to the definition of the Fourier coefficients $\widehat{f}(\omega)$ of an almost periodic function.

Definition 3.3. *If $\omega \in \mathbb{R}^n$ we set $\chi_\omega(x) = \exp(2\pi i \omega \cdot x)$ and the corresponding Fourier coefficient of f is defined by*

$$\widehat{f}(\omega) = \mathcal{M}(\overline{\chi}_\omega f) \tag{3.5}$$

where $\overline{\chi_\omega(x)}$ is the complex conjugate of $\chi_\omega(x)$.

The notation $\widehat{f}(\omega)$ can be confusing since $\widehat{f}(\omega)$ is not the value at ω of the distributional Fourier transform $\widehat{f}$ of f. This issue is discussed below (Theorem 3.8).

If f is almost periodic, so is $|f|^2$, and one has

$$\mathcal{M}(|f|^2) \;=\; \sum_\omega |\widehat{f}(\omega)|^2. \tag{3.6}$$

Therefore the set S of frequencies ω for which $\widehat{f}(\omega) \neq 0$ is at most a numerable set.

Definition 3.4. *The set* $S = \{\,\omega \in \mathbb{R}^n \mid \widehat{f}(\omega) \neq 0\,\}$ *is the spectrum of* f.

The Fourier series of f is the formal series:

$$f(x) \sim \sum_{\omega \in S} \widehat{f}(\omega)\, e^{2\pi i \omega \cdot x}. \tag{3.7}$$

This Fourier series (3.7) of f becomes an ordinary Fourier series if f is extended as a continuous function F on a suitable compact group $\mathcal{G}$ containing $\mathbb{R}^n$ as a dense subgroup.

Definition 3.5. *Let* $\Gamma \subset \mathbb{R}^n$ *be a subgroup of* $\mathbb{R}^n$. *A function* $\chi\colon \Gamma \to \mathbb{T}$ *is a weak character on* Γ *if it maps the additive group* Γ *to the multiplicative group* $\mathbb{T}$ *of complex numbers of modulus 1 and if it is a group homomorphism:*

$$\chi(x+y) = \chi(x)\chi(y) \;\; (\forall x, y \in \Gamma).$$

No continuity is required here. One can object that a strong character (i.e. a continuous one) is also a weak character. This terminology issue will not affect the proofs which are given in this essay. Let f be an almost periodic function and S be its spectrum. Let Γ be the additive subgroup of $\mathbb{R}^n$ generated by S. Let $\mathcal{G}$ be the compact group of all weak characters $\chi\colon \Gamma \to \mathbb{T}$. The topology on $\mathcal{G}$ is defined by the pointwise convergence on Γ of the corresponding weak characters.

Lemma 3.6. *The inclusion* $\Gamma \subset \mathbb{R}^n$ *yields the dual inclusion* $\mathbb{R}^n \subset \mathcal{G}$ *since every continuous character on* $\mathbb{R}^n$ *can be restricted to* Γ. *Then the almost periodic function* f *extends by continuity to* $\mathcal{G}$. *We denote by* F *the extension of* f *to* $\mathcal{G}$. *Conversely if* F *is a continuous function on* $\mathcal{G}$ *its restriction to* $\mathbb{R}^n$ *is an almost periodic function denoted by* f. *Finally the ordinary Fourier series of* F *on* $\mathcal{G}$ *coincides with the Fourier series of the almost periodic function* f.

One can ignore S at the expense of replacing $\mathcal{G}$ by a much larger compact group. The Bohr compactification of $\mathbb{R}^n$ is the compact group $\widetilde{\mathbb{R}}^n$ consisting of all weak characters $\chi\colon \mathbb{R}^n \to \mathbb{T}$. As above we have $\mathbb{R}^n \subset \widetilde{\mathbb{R}}^n$ and every almost periodic function f is the restriction to $\mathbb{R}^n$ of a continuous function on $\widetilde{\mathbb{R}}^n$.

If $\sum_{\omega \in S} |\widehat{f}(\omega)|$ is finite the Fourier series expansion of f converges to f uniformly on $\mathbb{R}^n$. Moreover the distributional Fourier transform of f is the atomic measure $\sum_{\omega \in S} \widehat{f}(\omega)\delta_\omega$ where δ_ω is the Dirac mass at ω. If $\sum_{\omega \in S} |\widehat{f}(\omega)|$ is infinite some summation procedures generalizing Cesaro summation are needed to give a meaning to (3.7). H. Bohr proved the following theorem:

Theorem 3.7. *Let* $S \subset \mathbb{R}^n$ *be a numerable set. For every* $\epsilon > 0$ *there exist a finite subset* $S(\varepsilon) \subset S$ *and a family* $\beta_S(\varepsilon, \omega)$, $\omega \in S$, *of weight factors with the following properties*

(*a*) $0 \le \beta_S(\varepsilon, \omega) \le 1$
(*b*) $\lim_{\varepsilon \downarrow 0} \beta_S(\varepsilon, \omega) = 1$ *for each* $\omega \in S$
(*c*) $\beta_S(\varepsilon, \omega) = 0$ *if* ω *does not belong to the finite set* $S(\varepsilon)$

(*d*) *For every almost periodic function f whose spectrum is contained in S we have* $\|f - P_\varepsilon\|_\infty \to 0$, $\varepsilon \to 0$, *when*

$$P_\varepsilon(x) \;=\; \sum_{\omega \in S(\varepsilon)} \beta_S(\varepsilon, \omega)\, \widehat{f}(\omega)\, \exp(2\pi i \omega \cdot x). \tag{3.8}$$

Is the distributional Fourier transform $\widehat{f}$ of an almost periodic function f given by the series $\sum_{\omega \in S} \widehat{f}(\omega)\delta_\omega$ where δ_ω is the Dirac mass at ω? This is not true at this naïve level. We cannot write $\widehat{f} = \sum_{\omega \in S} \widehat{f}(\omega)\delta_\omega$ since this sum of Dirac masses is not defined in general. If this sum is an atomic Radon measure then the two definitions of the Fourier transform of an almost periodic function agree as indicated in the following theorem:

Theorem 3.8. *Let f be an almost periodic function. Let us denote by S the spectrum of f and by $\widehat{f}(\omega)$, $\omega \in S$, the Fourier coefficients of f. Then the three following properties of f are equivalent*

(1) *For every $R \geq 1$ the sum $\sum_{\{\omega \in S \,|\, |\omega| \leq R\}} |\widehat{f}(\omega)|$ is finite.*
(2) *The distributional Fourier transform of f is a Radon measure.*
(3) *The distributional Fourier transform of f is the atomic measure $\sum_{\omega \in S} \widehat{f}(\omega)\delta_\omega$.*

The implication (1) $\Rightarrow$ (3) is interesting. Indeed the computation of the Fourier coefficients $\widehat{f}(\omega)$, $\omega \in S$, is often much easier than the determination of the distributional Fourier transform of f. Let ϕ be a function in the Schwartz class such that $\widehat{\phi}$ be compactly supported and $\widehat{\phi}(0) = 1$. Let $\phi_j = j^n \phi(jx)$, $j \in \mathbb{N}$. Let $f_j(x) = f * \phi_j(x)$. The Fourier coefficients of f_j are $\widehat{\phi}(\omega/j)\widehat{f}(\omega)$. Then (1) implies that $\sum_{\omega \in S} |\widehat{\phi}(\omega/j)\widehat{f}(\omega)|$ is finite. Therefore $f_j(x) = \sum_{\omega \in S} \widehat{\phi}(\omega/j)\widehat{f}(\omega)\exp(2\pi i \omega \cdot x)$ and the distributional Fourier transform $\widehat{f_j}$ is the atomic measure $\sum_{\omega \in S} \widehat{\phi}(\omega/j)\widehat{f}(\omega)\delta_\omega$. It suffices to let j tend to infinity to conclude.

The implication (2) $\Rightarrow$ (1) will follow from Lemma 3.9.

Lemma 3.9. *If the distributional transform of an almost periodic function f is a Radon measure μ then for every $\omega \in \mathbb{R}^n$ we have $\widehat{f}(\omega) = \mu(\{\omega\})$.*

The proof is straightforward. Let $\phi(x)$ be an even compactly supported test function such that $\phi(0) = 1$. Then $\mu(\{\omega\}) = \lim_{\epsilon \to 0} I(\epsilon)$ where $I(\epsilon) = \int \phi(\frac{x-\omega}{\epsilon})\, d\mu(x)$. But on the Fourier transforms side we have

$$I(\epsilon) = \epsilon^n \int f(x)\widehat{\phi}(\epsilon x)\exp(-2\pi i \omega \cdot x)\, dx$$

which tends to $\widehat{f}(\omega)$ since f is an almost periodic function.

Let us return to (2) $\Rightarrow$ (1). If μ is the distributional Fourier transform of f, Lemma 3.9 and (2) imply

$$\sum_{\{\omega \in S \,|\, |\omega| \leq R\}} |\widehat{f}(\omega)| \leq C_R < \infty$$

which ends the proof. Finally (3) $\Rightarrow$ (2) is obvious.

Definition 3.10. *The Banach algebra consisting of all almost periodic functions on $\mathbb{R}^n$ whose Fourier series is absolutely convergent will be denoted by $\mathcal{A}(\mathbb{R}^n)$.*

The definition of an almost periodic sequence will be needed in this essay.

Definition 3.11. *An almost periodic sequence c_k, $k \in \mathbb{Z}$, is the restriction to $\mathbb{Z}$ of an almost periodic function. An almost periodic sequence belongs to $\mathcal{A}(\mathbb{Z})$ if it is the restriction to $\mathbb{Z}$ of a function of $\mathcal{A}(\mathbb{R})$.*

An almost periodic sequence can also be defined by the natural generalization of (3.2).

4. Kahane's property

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set and let $\mathcal{T}_\Lambda$ be the vector space consisting of all trigonometric sums $P(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$ whose frequencies belong to Λ.

Definition 4.1. *Property $T(\Lambda)$ is satisfied if the space $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$.*

Property $T(\Lambda)$ is satisfied if and only if there exists a non trivial compactly supported Radon measure μ whose Fourier transform vanishes on Λ. This is the case if Λ is uniformly discrete or if Λ is a finite union of uniformly discrete sets. In the one dimensional case Arne Beurling and Paul Malliavin proved the equivalence between $T(\Lambda)$ and a finite density condition [1].

Definition 4.2. *If property $T(\Lambda)$ is satisfied, the closure of $\mathcal{T}_\Lambda$ in $\mathcal{C}(\mathbb{R}^n)$ is denoted by $\mathcal{C}_\Lambda$.*

Lemma 4.3. *A mean periodic function f whose spectrum is simple and contained in Λ belongs to $\mathcal{C}_\Lambda$ and the converse is true.*

This follows from Malgrange's theorem.

Lemma 4.4. *Let us assume that $T(\Lambda)$ is satisfied. Let $f \in \mathcal{C}(\mathbb{R}^n)$ with a polynomial growth at infinity. Let us assume that the distributional Fourier transform of f is a sum of weighted Dirac measures supported by Λ: $\widehat{f} = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$. Then $f \in \mathcal{C}_\Lambda$.*

The proof is easy. Let ϕ be a function in the Schwartz class whose Fourier transform $\widehat{\phi}$ is compactly supported and satisfies $\widehat{\phi}(0) = 1$. Let $\phi_j(x) = j^n\phi(jx)$. Then $f * \phi_j$ tends to f uniformly on compact sets as j tends to infinity. Here the polynomial growth of f at infinity is needed. But the Fourier transform of $f * \phi_j$ is a finite sum of Dirac measures. Therefore $f * \phi_j$ is a trigonometric polynomial and $f \in \mathcal{C}_\Lambda$. The converse implication is not true and a function $f \in \mathcal{C}_\Lambda$ does not have in general a polynomial growth at infinity. This fundamental problem is at the heart of this essay. We already mentioned the following definition introduced by Jean-Pierre Kahane in [3]:

Definition 4.5. *Let us assume that $T(\Lambda)$ is satisfied. Property $Q(\Lambda)$ holds if every $f \in \mathcal{C}_\Lambda$ is an almost periodic function in the sense of Harald Bohr.*

An equivalent definition of the property $Q(\Lambda)$ was given by Kahane in [3]:

Lemma 4.6. *$\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. Then $Q(\Lambda)$ is equivalent to the following condition: there exist a compact set K and a constant C such that for every finite trigonometric sum*

$$f(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x)$$

whose frequencies belong to Λ one has

$$\|f\|_\infty \leq C \sup_{x \in K} |f(x)|. \tag{4.1}$$

This was our first problem about trigonometric sums. A dual version of Lemma 4.6 is the following:

Lemma 4.7. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. Then $Q(\Lambda)$ is equivalent to the following condition: there exist a compact set K and a constant C such that for every $x_0 \in \mathbb{R}^n$ there exists a complex Radon measure μ_{x_0} enjoying the following three properties*

(i) μ_{x_0} is supported by K
(ii) the total mass $\|\mu_{x_0}\|$ of μ_{x_0} does not exceed C
(iii) $\widehat{\mu}_{x_0}(\lambda) = \exp(2\pi i x_0 \cdot \lambda)$, $\forall \lambda \in \Lambda$.

We now turn to our second problems about trigonometric sums.

Definition 4.8. *Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. Then Λ is mild if there exist a continuous weight function $\omega(x) \geq 1$ with a polynomial growth at infinity and a compact set $K \subset \mathbb{R}^n$ such that for every trigonometric sum $P \in \mathcal{T}_\Lambda$ and every $x \in \mathbb{R}^n$ we have*

$$|P(x)| \leq \omega(x) \sup_{y \in K} |P(y)|. \tag{4.2}$$

This definition will be slightly modified in Section 9, Definition 9.2. This can be given a simpler formulation as the following theorem shows:

Theorem 4.9. *Let $\omega \geq 1$ be a continuous function defined on $\mathbb{R}^n$. Then the following properties of a locally finite set Λ are equivalent ones:*

(a) Every mean periodic function $\mathcal{C}_\Lambda$ is $0(\omega)$ at infinity.
(b) There exist a compact set K and a constant C such that for every $P \in \mathcal{T}_\Lambda$ and every $x \in \mathbb{R}^n$ one has:

$$|P(x)| \leq \omega(x) \sup_{y \in K} |P(y)|. \tag{4.3}$$

The implication $(b) \Rightarrow (a)$ is trivial since (4.3) extends by continuity to every $f \in \mathcal{C}_\Lambda$. It remains to prove that $(a) \Rightarrow (b)$. If (b) is not satisfied for every $T > 1$, $M > 1$, $\epsilon > 0$ there exist a $P \in \mathcal{T}_\Lambda$ and a $x \in \mathbb{R}^n$ such that

$$\sup_{|x| \leq T} |P(x)| \leq \epsilon, \quad |P(y)| \geq M\omega(x). \tag{4.4}$$

Then we define $\mathcal{C}_\Lambda$ by $f = \sum_0^\infty P_j$ where P_j satisfies (4.4) for a suitable sequence (ϵ_j, T_j, M_j), $j \in \mathbb{N}$. It suffices to impose $\epsilon_j = 2^{-j}$, $T_{j+1} \geq T_j + |y_0| \cdots + |y_j|$ and $M_{j+1} \geq 2^j(1 + \|P_0\|_\infty + \cdots + \|P_j\|_\infty) - \sum_{j+2}^\infty \epsilon_m$. Then

$$|f(y_{j+1})| \geq M_j\omega(y_{j+1}) - \sum_0^j \|P_m\|_\infty - \sum_{j+2}^\infty \epsilon_m \geq (M_{j+1}-1)\omega(y_{j+1}) \tag{4.5}$$

which ends the proof.

Let $\alpha > 0$ be irrational. A nontrivial example of a uniformly discrete mild set is

$$\Lambda_\alpha = -\mathbb{N} \cup \alpha\mathbb{N} = \{\ldots, -3, -2, -1, 0, \alpha, 2\alpha, 3\alpha, \ldots\}. \tag{4.6}$$

This set does not satisfy Kahane's condition (Theorem 7.7). However Λ_α is mild and every $f \in \mathcal{C}_{\Lambda_\alpha}$ is $O(\sqrt{|x|})$ at infinity. This will be proved in Section 9. More generally a gentle set (Definition 5.18 below) is mild (Theorem 9.3). But there exist some uniformly discrete sets Λ for which (4.3) does not hold whatever be the compact set K and the weight ω. If Λ is not uniformly discrete Kahane's property $Q(\Lambda)$ cannot be satisfied. However (4.3) makes sense.

Let $\alpha > 0$, $\alpha \notin \mathbb{Q}$, $\beta > 0$, and $\lambda_k^{(\alpha,\beta)} = k + \beta\sin(2\pi\alpha k)$, $k \in \mathbb{Z}$. Let

$$\Lambda_{\alpha,\beta} = \{\, \lambda_k^{(\alpha,\beta)} \mid k \in \mathbb{Z}\,\}. \tag{4.7}$$

The content of *Theorem 6.3* of [7] is that Kahane's property $Q(\Lambda_{\alpha,\beta})$ does not hold. In other terms we have:

Theorem 4.10. *Let* $\Lambda_{\alpha,\beta}$ *be defined by* (4.7)*. Then there exists a mean periodic function* $\boldsymbol{g}$ *whose spectrum is simple and contained in* $\Lambda_{\alpha,\beta}$ *and which is not almost periodic.*

But it will be proved in Section 9 that the set $\Lambda_{\alpha,\beta}$ is mild and that every $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ is $O(\sqrt{|x|})$ at ∞.

Here is the first half of the proof of Theorem 4.10. If $\beta\,|\sin(\pi\alpha)| \geq 1/2$, $\Lambda_{\alpha,\beta}$ is not uniformly discrete and $Q(\Lambda_{\alpha,\beta})$ does not hold. Let us assume $\beta\,|\sin(\pi\alpha)| < 1/2$ and prove Theorem 4.10. The function $\boldsymbol{g}$ is not constructed explicitly in this essay but instead an indirect approach is used. Let δ_a be the Dirac measure at a and let us consider the atomic measure

$$\sigma_{\alpha,\beta} = \sum_{-\infty}^{\infty} \delta_{\lambda_k^{(\alpha,\beta)}}. \tag{4.8}$$

This measure $\sigma_{\alpha,\beta}$ is an almost periodic measure (Lemma 5.14). Moreover its distributional Fourier transform $\widehat{\sigma}_{\alpha,\beta}$ is also an atomic measure (Theorem 5.16). But $\widehat{\sigma}_{\alpha,\beta}$ is not a translation bounded measure (Theorem 5.16). This immediately follows from Paul Cohen's theorem (Theorem 5.17).

We then argue by contradiction. If Kahane's property $Q(\Lambda_{\alpha,\beta})$ was satisfied then every measure μ whose spectrum is contained in $\Lambda_{\alpha,\beta}$ would be an almost periodic measure (Proposition 5.9). This is not the case since $\widehat{\sigma}_{\alpha,\beta}$ is not an almost periodic measure (it is not even translation bounded). This proof is not complete since Proposition 5.9 and Theorem 5.16 are not yet proved. This will be achieved in sections 5 and 6. Then the proof of Theorem 4.10 will be complete. The main ingredients of this proof are (a) the study of almost periodic measures and (b) the computation of the distributional Fourier transform of an almost periodic measure.

Kahane's property $Q(\Lambda_{\alpha,\beta})$ does not hold. Moreover the weaker property $Q(\Lambda_{\alpha,\beta}, 1)$ investigated by Kahane in [3] does not hold either. We do not know if $Q(\Lambda_{\alpha,\beta}, p)$ holds for $p \neq 2$. Property $Q(\Lambda, p)$ is defined in Section 8.

5. Almost periodic measures

The tools needed to prove that $\widehat{\sigma}_{\alpha,\beta}$ is an atomic measure are developed in this section. Let $\mathcal{D}$ denote the space of infinitely differentiable functions with compact support. Laurent Schwartz proposed the following definition of an almost periodic distribution.

Definition 5.1. *A distribution S is almost periodic if for every testing function $\phi \in \mathcal{D}$ the convolution product $S * \phi$ is an almost periodic function in the sense given by Bohr.*

This naturally extends to almost periodic measures. The class $\mathcal{D}$ of testing functions is replaced by the class $\mathcal{E}$ of compactly supported continuous functions.

Definition 5.2. *A Radon measure μ on $\mathbb{R}^n$ is almost periodic if for every compactly supported continuous function g the convolution product $\mu * g = f$ is an almost periodic function in the sense given by Bohr.*

If μ is an almost periodic measure the closed graph theorem implies the following:

$$\sup_{x \in \mathbb{R}^n} \int_{B(x)} |d\mu| < \infty. \tag{5.1}$$

Here $B(x)$ is the ball centered at x with radius 1. We say that μ is a *translation bounded measure* and denote by $\|\mu\|_*$ the norm defined by the LHS of (6.1). We then have:

Lemma 5.3. *A translation bounded measure μ is an almost periodic measure if and only if μ is an almost periodic distribution.*

The Fourier coefficients of an almost periodic measure μ are defined as follows. Let g be a compactly supported continuous function such that $\int g(x)\,dx = 1$. Then $\mu * g$ is an almost periodic function.

Lemma 5.4. *We set $\widehat{\mu}(0) = \mathcal{M}(\mu * g)$ and this does not depend on g.*

If μ is an almost periodic measure and if f is an almost periodic function then the product $f\mu$ is an almost periodic measure. If $\omega \in \mathbb{R}^n$ we set $\chi_\omega(x) = \exp(2\pi i\omega \cdot x)$ and the Fourier coefficients of μ are defined by $\widehat{\mu}(\omega) = \mathcal{M}(\overline{\chi}_\omega \mu)$. The *spectrum* of μ is the set $S = \{\, \omega \mid \widehat{\mu}(\omega) \neq 0 \,\}$.

Moreover an almost periodic measure μ is a continuous linear form on the Banach space of almost periodic functions. The pairing is defined by $\langle \mu, f \rangle = \mathcal{M}(f\mu)$. Therefore μ defines a Radon measure $\tilde{\mu}$ on the Bohr compactification $\widetilde{\mathbb{R}^n}$ of $\mathbb{R}^n$ and we have $\widehat{\mu}(\omega) = \widehat{\tilde{\mu}}(\omega)$, $\forall \omega \in \mathbb{R}^n$. This remark will be used later on.

Theorem 3.8 extends to almost periodic measures.

Theorem 5.5. *Let σ be an almost periodic measure. Let us denote by S the spectrum of σ and by $\widehat{\sigma}(\omega)$, $\omega \in S$, the Fourier coefficients of σ. Then the three following properties of σ are equivalent*

(1) *For every $R \geq 1$ the sum $\sum_{\{\omega\in S||\omega|\leq R\}}|\widehat{\sigma}(\omega)|$ is finite.*
(2) *The distributional Fourier transform of σ is a Radon measure.*
(3) *The distributional Fourier transform of σ is the atomic measure $\sum_{\omega\in S}\widehat{\sigma}(\omega)\delta_\omega$.*

To prove Theorem 5.5 it suffices to apply Theorem 3.8 to $f = \sigma * \phi$ where ϕ is a test function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Definition 5.6. *A tempered distribution σ is a Poisson measure if both σ and its distributional Fourier transform $\widehat{\sigma}$ are atomic measures.*

In a preliminary version of this essay a Poisson measure was defined as an almost periodic measure whose distributional Fourier transform is also an almost periodic measure. Let σ be a Poisson measure. Then $\sigma = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\widehat{\sigma} = \sum_{s\in S} b(s)\delta_s$. This implies the following variant of the Poisson summation formula

$$\sum_{\lambda\in\Lambda} a(\lambda)\widehat{f}(\lambda) = \sum_{s\in S} b(s)f(s) \tag{5.2}$$

where f is a test function.

Lemma 5.7. *Let us assume that both σ and $\widehat{\sigma}$ are almost periodic measures. Then σ is a Poisson measure.*

This follows from Theorem 5.5.

Lemma 5.8. *Let σ be a translation bounded Poisson measure. Then σ is an almost periodic measure.*

Let ϕ be a test function in the Schwartz class whose Fourier transform $\widehat{\phi}$ is compactly supported. Then the product $\widehat{\sigma}\widehat{\phi}$ has a finite total mass. Therefore $\sigma * \phi$ is an almost periodic function. Let g be a compactly supported continuous function. There exists a sequence ϕ_j tending to g for the norm $\sum_{k\in\mathbb{Z}^n}\sup_{|x-k|\leq 1}|f(x)|$. Since σ is translation bounded we can pass to the limit and $\sigma * g$ is an almost periodic.

We then have

Proposition 5.9. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set and let us assume that $Q(\Lambda)$ holds. Let μ be a Radon measure such that (a) μ is a tempered distribution and (b) the distributional Fourier transform $\widehat{\mu}$ of μ is an atomic measure supported by Λ. Then μ is an almost periodic measure.*

Before proving this fact let us stress that it does not characterize Kahane's $Q(\Lambda)$ property. A counter example is given in Section 8. The proof of Proposition 5.9 begins with the following lemma:

Lemma 5.10. *If g is a compactly supported continuous function then $f = \mu * g$ is a mean periodic function whose spectrum is simple and contained in Λ.*

If Lemma 5.10 is accepted then property $Q(\Lambda)$ implies that f is an almost periodic function which ends the proof of Proposition 5.9. Let us prove Lemma 5.10. Since Λ is uniformly discrete there exists a non trivial compactly supported $h \in L^2(\mathbb{R}^n)$ such that $\widehat{h} = 0$ on Λ [4]. Then $f * h = \mu * g * h$ and the distributional Fourier transform of $f * h$

is the product $\widehat{\mu}\,\widehat{g}\,\widehat{h}$. But $\widehat{\mu}$ is a sum of Dirac measures on Λ. Therefore $\widehat{\mu}\,\widehat{h} = 0$ and $f * h = 0$. The function f satisfies a non trivial convolution equation $f * h = 0$ where h is compactly supported. Then f is a mean periodic function. The Fourier transform of $\mu * g$ is the product $\widehat{\mu}\,\widehat{g}$. Therefore $\widehat{f}$ is an atomic measure supported by Λ and the spectrum of $\mu * g$ is simple. Lemma 5.10 is proved.

Lemma 5.11. *Let σ be an almost periodic Poisson measure. Then for $\omega \in \mathbb{R}^n$ we have*

$$\widehat{\sigma}(\omega) = \widehat{\sigma}(\{\omega\}). \tag{5.3}$$

This lemma looks tautological but is not. In the left-hand side we meet a Fourier coefficient of an almost periodic measure and in the right-hand side a mass of an atomic measure. However Lemma 5.11 is an easy corollary of Lemma 3.9.

Theorem 5.12. *Let $\Lambda \subset \mathbb{R}$ a closed and discrete set and let $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$. Then the following two properties of Λ are equivalent*

(*a*) *$\widehat{\sigma}_\Lambda$ is an almost periodic measure*

(*b*) *$\Lambda = \bigcup_1^N (\alpha_j\mathbb{Z} + \beta_j)$ where $\alpha_j > 0, \beta_j \in \mathbb{R}$.*

The implication $(b) \Rightarrow (a)$ is trivial. Let us prove $(a) \Rightarrow (b)$. Let μ be the inverse Fourier transform of σ_Λ. Then μ defines a Radon measure $\tilde{\mu}$ on the Bohr compactification $\widetilde{\mathbb{R}}$ of $\mathbb{R}$. By Lemma 5.11 the Fourier coefficients of $\tilde{\mu}$ are the masses of σ_Λ. But these masses are 0 or 1. Therefore $\tilde{\mu}$ is an idempotent measure on $\widetilde{\mathbb{R}}$. Paul Cohen's theorem ends the proof (see also Theorem 5.17 below).

The definition of almost periodic measures is very demanding. If $\Lambda \subset \mathbb{R}^n$ is a model set which is not a lattice [8] then $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is not an almost periodic measure. This was observed by J. Lagarias. For example let $\lfloor x \rfloor$ be the integral part of x and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Let α be an irrational number and let

$$\lambda_k = k + \{\alpha k\}, \quad k \in \mathbb{Z}. \tag{5.4}$$

Then

$$\sigma_\Lambda = \sum_{k\in\mathbb{Z}} \delta_{\lambda_k} \tag{5.5}$$

is not an almost periodic measure.

The conclusion changes dramatically if the sawtooth function $x \mapsto \{x\}$ is replaced by an almost periodic function $\theta \in \mathcal{A}$.

Definition 5.13. *We now assume that $\lambda_k = k + \theta(k)$, $k \in \mathbb{Z}$, where $\theta \in \mathcal{A}(\mathbb{Z})$ (Definition 3.6). Let*

$$\sigma_\theta = \sum_{k\in\mathbb{Z}} \delta_{\lambda_k}. \tag{5.6}$$

Then Λ_θ is the support of the measure σ_θ.

The mapping $\theta \mapsto \sigma_\theta$ will be studied in this essay. The measure σ_θ is not given in general by $\sum_{\lambda\in\Lambda_\theta} \delta_\lambda$ since $\lambda_k = \lambda_l$, $k \neq l$, can happen. We have $\sigma_\theta = \sum_{\lambda\in\Lambda_\theta} m(\lambda)\delta_\lambda$ where the multiplicities $m(\lambda)$ satisfy $m(\lambda) \in \mathbb{N}$ and $1 \leq m(\lambda) \leq C$ for some constant C.

Lemma 5.14. *Let $\theta(k)$, $k \in \mathbb{Z}$ be an almost periodic sequence. Then the measure σ_θ is an almost periodic measure.*

Then let ϕ be a compactly supported continuous function and let $f(x) = \sum \phi(x - k - \theta(k))$. We need to show that $f(x)$ is an almost periodic function. If $\tau \in \mathbb{Z}$ we obviously have

$$f(x + \tau) = \sum_{k \in \mathbb{Z}} \phi(x - k - \theta(k + \tau)). \tag{5.7}$$

But θ is an almost periodic sequence. Therefore there exists a relatively dense set of integers M_ϵ such that $|\theta(k + \tau) - \theta(k)| \leq \epsilon$ uniformly in k. It now suffices to observe that the series defining $f(x)$ is locally finite. This concludes the proof. The following lemma will be needed in this essay.

Lemma 5.15. *If f is an almost periodic function we have*

$$\mathcal{M}(f\sigma_\theta) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{-N}^{N} f(\lambda_k). \tag{5.8}$$

Finally we have:

Theorem 5.16. *Let us assume $\theta \in \mathcal{A}(\mathbb{Z})$. Then the almost periodic measure σ_θ defined by* (5.6) *is a Poisson measure. However $\widehat{\sigma}_\theta$ is not a translation bounded measure unless $\theta(k)$, $k \in \mathbb{Z}$, is a periodic sequence.*

The first assertion still holds if σ_θ is replaced by $\tau_\theta = \sum_{k \in \mathbb{Z}} c(k)\delta_{\lambda_k}$ and if the sequence c belongs to $\mathcal{A}(\mathbb{Z})$. The first assertion in Theorem 5.16 will be proved in Section 6. Let us consider the second assertion. If $\|\theta\|_\infty < 1/2$ we have $\lambda_k \neq \lambda_l$ if $k \neq l$. Then Theorem 5.12 and Lemma 5.8 imply that $\widehat{\sigma}_\theta$ cannot be an almost periodic measure. This simple proof shall be slightly modified if multiplicities occur in Λ_θ. Here is the argument. As it was observed above $\sigma_\theta = \sum_{\lambda \in \Lambda_\theta} m(\lambda)\delta_\lambda$ where the multiplicities $m(\lambda) \in \mathbb{N}$ belong to $[1, C]$ for some constant C. We then use the following result (Theorem 4.10 of [5]).

Theorem 5.17. *Let $\Lambda \subset \mathbb{R}$ be a closed and discrete set and let us assume that the complex numbers $m(\lambda)$, $\lambda \in \Lambda$, belong to a finite set $F \subset \mathbb{C} \setminus \{0\}$. Let σ be the Radon measure $\sigma = \sum_{\lambda \in \Lambda} m(\lambda)\delta_\lambda$. Let us assume that the distributional Fourier transform $\widehat{\sigma}$ is a Radon measure μ which satisfies for every $R \geq 1$*

$$|\mu|([-R, R]) \leq CR. \tag{5.9}$$

Then

$$\Lambda = F \bigtriangleup \bigcup_{1}^{N} (\alpha_j \mathbb{Z} + \beta_j) \tag{5.10}$$

where $\alpha_j > 0$, $\beta_j \in \mathbb{R}$, F is finite and $\bigtriangleup$ denotes the symmetrical difference.

But Λ_θ cannot be a finite union of arithmetical progressions up to a finite set unless θ is a periodic sequence. Therefore $\widehat{\sigma}_\theta$ cannot be a translation bounded measure unless θ is a periodic sequence.

The following definition is used in Section 7.

Definition 5.18. *A uniformly discrete set $\Lambda \subset \mathbb{R}^n$ is a gentle set if there exist a uniformly discrete set M containing Λ and an atomic measure $\sigma = \sum_{m\in M} c(m)\delta_m$ such that:*

(a) $\Lambda = \{ m \in M \mid c(m) = 1 \}$
(b) *The distributional Fourier transform $\widehat{\sigma}$ of σ is a Radon measure.*

The class of gentle sets contains lattices and quasi-crystals. In the following section it will be proved that $\Lambda_{\alpha,\beta}$ is a gentle set. More generally Λ_θ (Definition 5.13) is a gentle set. In Section 9 it will be proved (Theorem 9.3) that most gentle sets are mild sets (Definition 4.8). We show in Section 7 that a gentle set Λ satisfies Kahane's property $Q(\Lambda)$ whenever $\widehat{\sigma}$ is a translation bounded Radon measure (Theorem 7.1).

6. End of the proof of Theorem 5.16

This proof relies on Theorem 3.8. We already know that σ_θ is an almost periodic measure. Instead of computing the distributional Fourier transform of σ_θ, which is not easy, we compute the Fourier coefficients of the almost periodic measure σ_θ and then check condition (1) of Theorem 3.8.

For every real number ω we compute the Fourier coefficient $\widehat{\sigma}_\theta(\omega)$ of the almost periodic measure σ_θ. Lemma 5.15 implies that it suffices to average over $\mathbb{Z}$. From now on the operator $\mathcal{M}$ denotes the mean value over $\mathbb{Z}$. Then we have:

$$\widehat{\sigma}_\theta(\omega) = \mathcal{M}\big[\exp\big(-2\pi i\omega(k+\theta(k))\big)\big]. \tag{6.1}$$

But

$$\exp\big(-2\pi i\omega(k+\theta(k))\big) = \exp(-2\pi i\omega k)\sum_0^\infty \frac{(-2\pi i\omega)^m}{m\,!}(\theta(k))^m \tag{6.2}$$

and the RHS converges uniformly in k. Taking the mean values with respect to k yields

$$\mathcal{M}\big[\exp\big(-2\pi i\omega(k+\theta(k))\big)\big] = \sum_0^\infty \frac{(-2\pi i\omega)^m}{m\,!}\widehat{\theta^m}(\omega). \tag{6.3}$$

We have $\sum_\omega |\widehat{\theta^m}(\omega)| \le \|\theta\|_{\mathcal{A}}^m$ since $\mathcal{A}$ is a Banach algebra. Finally

$$\begin{aligned}\sum_{|\omega|\le R} |\widehat{\sigma}_\theta(\omega)| &\le \sum_0^\infty \frac{(2\pi R)^m}{m\,!}\sum_{|\omega|\le R}|\widehat{\theta^m}(\omega)| \\ &\le \sum_0^\infty \frac{(2\pi R)^m}{m\,!}\|\theta\|_{\mathcal{A}}^m = \exp(2\pi R\|\theta\|_{\mathcal{A}}).\end{aligned}$$

Corollary 6.1. *The distributional Fourier transform $\widehat{\sigma}_\theta$ of the almost periodic measure σ_θ is the atomic measure $\sum_\omega \widehat{\sigma}_\theta(\omega)\delta_\omega$.*

Corollary 6.1 implies the first assertion in Theorem 5.16. The second assertion was already proved. Therefore the proof of Theorem 5.16 is complete.

Let S the spectrum of θ and $\Gamma \subset \mathbb{R}$ be the additive subgroup generated by S. Let G be the compact group which is the dual of Γ. Then θ extends continuously to G. By an abuse of notations this extension is also denoted by θ. Then (6.3) can be rewritten as

$$\int_G \exp\bigl[-2\pi i x(y+\theta(y))\bigr]\,dy = \sum_0^\infty \frac{(-2\pi i x)^m}{m!}\int_G \exp(-2\pi i x y)\theta^m(y)\,dy. \tag{6.4}$$

Both sides of (6.4) vanish if $x \notin \Gamma$. Therefore we shall have $x \in \Gamma$ in (6.4) and the support of $\widehat{\sigma}_\theta$ is contained in Γ.

Here is an example of the identity (6.4). Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the torus and let us assume that $\theta\colon \mathbb{T}^n \to \mathbb{R}$ belongs to the Wiener algebra $A(\mathbb{T}^n)$. Let $\alpha_1, \ldots, \alpha_n$ be n real numbers such that $1, \alpha_1, \ldots, \alpha_n$, are linearly independent over $\mathbb{Q}$. Let $\lambda_k = k + \theta(\alpha_1 k, \ldots, \alpha_n k)$ and $\sigma_\theta = \sum_{k\in\mathbb{Z}} \delta_{\lambda_k}$.

Proposition 6.2. *The distributional Fourier transform of σ_θ is the atomic measure defined by*

$$\widehat{\sigma}_\theta = \sum_{p\in\mathbb{Z}^n}\sum_{q\in\mathbb{Z}} \gamma(p, q+\alpha\cdot p)\delta_{q+\alpha\cdot p} \tag{6.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and

$$\gamma(p, q+\alpha\cdot p) = \int_{\mathbb{T}^n} \exp\bigl[-2\pi i\bigl(p\cdot u + (q+\alpha\cdot p)\theta(u)\bigr)\bigr]\,du. \tag{6.6}$$

Theorem 6.3. *Let us assume that the real valued function θ belongs to the Wiener algebra $\mathcal{A}(\mathbb{Z})$. Then $\Lambda_\theta = \{\, \lambda_k = k + \theta(k) \mid k \in \mathbb{Z} \,\}$ does not satisfy Kahane's property $Q(\Lambda)$ unless the sequence $\theta(k)$, $k \in \mathbb{Z}$, is periodic.*

Theorem 6.3 implies Theorem 4.10. Theorem 6.3 is an obvious consequence of Theorem 5.16 as it will be shown now. Reasoning by contradiction let us assume that $Q(\Lambda)$ holds. Then $Q(-\Lambda)$ holds as well. Proposition 5.9 applied to the Radon measure $\mu = \widehat{\sigma}_\theta$ would imply that μ is an almost periodic measure. We already know (Theorem 5.16) that it is not the case unless θ is a periodic sequence. Therefore Λ_θ does not satisfy Kahane's $Q(\Lambda)$ property unless θ is a periodic sequence. It was announced for the special case $\theta(k) = \sin(2\pi\alpha k)$ and $\alpha \notin \mathbb{Q}$ in [7].

7. Gentle sets

Some gentle sets satisfy Kahane's property as the following theorem shows.

Theorem 7.1. *Let $\sigma = \sum_{m\in M} c(m)\delta_m$ be an atomic measure supported by a uniformly discrete set $M \subset \mathbb{R}^n$. Let $\Lambda = \{\, m \in M \mid c(m) = 1 \,\}$. Let us assume that the distributional Fourier transform $\widehat{\sigma}$ of σ is a translation bounded Radon measure. Then Λ satisfies Kahane's property $Q(\Lambda)$.*

We cannot have $M = \Lambda$ unless Λ is a finite union of lattices up to a finite set (Theorem 5.17). The proof of Theorem 7.1 uses a characterization of Kahane's property given by Lemma 7.2. For a closed set $E \subset \mathbb{R}^n$ let $B(E)$ denote the Wiener algebra consisting of all restrictions to E of Fourier–Stieltjes transforms of bounded complex Radon measures on $\mathbb{R}^n$ [10], the norm in $B(E)$ being the quotient norm.

Lemma 7.2. *Let Λ be uniformly discrete. Then the following two properties are equivalent ones*

(*a*) *$Q(\Lambda)$ holds*

(*b*) *There exist a neighborhood V of 0 and a constant C such that (1) the sets $V+\lambda$, $\lambda \in \Lambda$, are pairwise disjoint and (2) for every $y \in \mathbb{R}^n$ the function $F_y(x)$ defined on $E = \Lambda + V$ by $F_y(\lambda + s) = \exp(2\pi i \lambda \cdot y)$, $\lambda \in \Lambda, s \in V$, satisfies $\|F_y\|_{B(E)} \leq C$.*

This is proved in [8], Theorem 6.6. Let us prove Theorem 7.1. Let B be a ball of radius $\beta > 0$ centered at 0 and such that the translated balls $m + B$, $m \in M$, are pairwise disjoint. Let V be a smaller ball of radius $\alpha < \beta$ centered at 0. Let ϕ be an even $\mathcal{C}^\infty$ function such that $\phi = 1$ on V and $\phi = 0$ outside B. Let $E = \Lambda + V$. We consider $F_y = (\chi_y \sigma) * \phi$. Then $\widehat{F_y} = \mu_y \widehat{\phi}$ where μ is the inverse Fourier transform of σ and μ_y is μ translated by y. Since μ is translation bounded the total mass of $\mu_y \widehat{\phi}$ does not exceed a constant C. Therefore $\|F_y\|_{B(E)} \leq C$. But $F_y(\lambda + s) = \chi_y(\lambda)$, $\forall \lambda \in \Lambda$, $\forall s \in V$. Then Lemma 7.2 concludes the proof.

Theorem 7.3. *A model set satisfies Kahane's property.*

This is known from [9] but the proof which is given here is new. The proof of Theorem 7.3 relies on the following lemma:

Lemma 7.4. *Let Λ be a model set. There exist a model set Λ' containing Λ and a Poisson measure $\sigma = \sum_{\lambda \in \Lambda'} c(\lambda)\delta_\lambda$ such that $c(\lambda) = 1$, $\forall \lambda \in \Lambda$. Moreover $\widehat{\sigma}$ is also an almost periodic measure.*

The proof relies on the definition of a model set which is given now for the reader's convenience. Let $m, n \in \mathbb{N}$, $N = m + n$, and $\Gamma \subset \mathbb{R}^N$ be a lattice: $\Gamma = A(\mathbb{Z}^N)$ where $A \in \mathbb{G}L_N(R)$. For $(x,t) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, we write $p_1(x,t) = x$, $p_2(x,t) = t$. Let us assume that p_1 once restricted to Γ is a 1–1 mapping with a dense range. The same is required from p_2.

Definition 7.5. *Let $I \subset \mathbb{R}^m$ be a Riemann integrable compact set (a window) with a positive measure. Then the model set $\Lambda_I \subset \mathbb{R}^n$ is defined by*

$$\Lambda_I = \{\, p_1(\gamma) \mid \gamma \in \Gamma,\, p_2(\gamma) \in I\,\}. \tag{7.1}$$

To prove Lemma 7.4 it suffices to introduce a larger window J which contains a neighborhood of I and an even $\mathcal{C}^\infty$ function w such that $w = 1$ on I and $w = 0$ outside J. We now prove that the atomic measure

$$\sigma = \sum_{\gamma \in \Gamma} w(p_2(\gamma))\delta_{p_1(\gamma)} \tag{7.2}$$

is a Poisson measure. Let ϕ be a test function in the Schwartz class. Let us compute $\langle \sigma, \widehat{\phi} \rangle = \sum_{\gamma \in \Gamma} w(p_2(\gamma))\widehat{\phi}(p_1(\gamma))$. Poisson summation formula applied to the lattice Γ and to the dual lattice Γ^* yields

$$\langle \sigma, \widehat{\phi} \rangle = c_\Gamma \sum_{\gamma^* \in \Gamma^*} \widehat{w}(p_2(\gamma^*))\phi(-p_1(\gamma^*)) = \langle \widehat{\sigma}, \phi \rangle. \tag{7.3}$$

We have $\widehat{\sigma} = c_\Gamma \sum_{\gamma^* \in \Gamma^*} \widehat{w}(-p_2(\gamma^*))\delta_{p_1(\gamma^*)}$. Therefore $\widehat{\sigma}$ is a translation bounded atomic measure. Finally σ is a Poisson measure which ends the proof of Lemma 7.4 and of Theorem 7.3.

Let $\theta \geq 3$ be a real number, let $q \geq 1$ be an integer, and let Λ^q be the set of all real numbers of the form $\lambda = \pm\theta^{j_1} \pm \theta^{j_2} \pm \cdots \pm \theta^{j_q}$ where the exponents $0 \leq j_1 < j_2 < \cdots < j_q$ are arbitrary integers. We then have:

Theorem 7.6. *For every integer $q \geq 1$ the set Λ^q satisfies Kahane's property.*

Let us observe that $\Lambda^q \subset \Lambda^{q+1}$ and the union $\cup \Lambda^q$ satisfies Kahane's property if and only if θ is a Pisot–Thue–Vijayaraghavan number. Let us sketch the proof of Theorem 7.6. The details will appear in a forthcoming paper.

We consider the finite product $P_j(x) = \prod_0^j (1 + \cos(2\pi\theta^k x))$ and set $\mu_j = P_j\, dx$. Then μ_j converges weakly to the Riesz product μ. This Riesz product is an almost periodic measure and the distributional Fourier transform of μ is an atomic measure σ to which Theorem 7.1 can be applied. The support M of σ is the set of all finite sums $m = \pm\theta^{j_1} \pm \theta^{j_2} \pm \cdots \pm \theta^{j_q}$ where $0 \leq j_1 < j_2 < \ldots < j_q$ and q is now an arbitrary integer. We then have $\sigma(\{\lambda\}) = 2^{-q}$, $\lambda \in \Lambda^q$. This implies that Theorem 7.1 shall be applied to the measure $2^q\sigma$.

This section is concluded with the proof of a theorem which was announced in [7]. If $\alpha \notin \mathbb{Q}$, $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$ cannot satisfy Kahane's property since Λ is not uniformly discrete. Is it the only obstruction? Let us assume that $M \subset \mathbb{Z} \cup \alpha\mathbb{Z}$ is uniformly discrete. Does $Q(M)$ hold ? A counter example is given by Theorem 7.7.

Theorem 7.7. *Let $\alpha > 0$ be a real number. Then*

$$\Lambda_\alpha = -\mathbb{N} \cup \alpha\mathbb{N} = \{\ldots, -4, -3, -2, -1, 0, \alpha, 2\alpha, 3\alpha, 4\alpha, \ldots\}$$

satisfies Kahane's $Q(\Lambda)$ property if and only if $\alpha \in \mathbb{Q}$.

If $\alpha \in \mathbb{Q}$ then Λ_α is contained in a lattice and trivially satisfies Kahane's property. If $\alpha \notin \mathbb{Q}$ Theorem 7.7 follows from a more general fact. The characterization of the subsets Λ of $\mathbb{Z} \cup \alpha\mathbb{Z}$ which satisfy Kahane's condition $Q(\Lambda)$ is given by the following theorem:

Theorem 7.8. *Let us assume that two uniformly discrete sets $\Lambda_1 \subset \mathbb{R}^n$ and $\Lambda_2 \subset \mathbb{R}^n$ both satisfy Kahane's property $Q(\Lambda)$. If there exists a complex Radon measure τ with a finite total mass such that $\widehat{\tau} = 1$ on Λ_1 and $\widehat{\tau} = 0$ on Λ_2, then $\Lambda = \Lambda_1 \cup \Lambda_2$ still satisfies $Q(\Lambda)$.*

The converse implication is true in one dimension when $\Lambda_1 \subset \mathbb{Z}$, $\Lambda_2 \subset \alpha\mathbb{Z}$, and $\alpha \notin \mathbb{Q}$: if $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfies Kahane's condition $Q(\Lambda)$ then there exists a complex Radon measure τ with a finite total mass such that $\widehat{\tau} = 1$ on Λ_1 and $\widehat{\tau} = 0$ on Λ_2.

The proof of the first assertion is not difficult. Let P be a trigonometric sum whose frequencies belong to Λ. Then $P = P_1 + P_2$ where the frequencies of P_1 belong to Λ_1 and the frequencies of P_2 belong to Λ_2. We have $P_1 = P * \tau$. But $\tau = \sigma + \rho$ where the total mass of ρ does not exceed ϵ and σ is supported by a compact set L. But $P_1 = P * \tau = P' + P''$, $P' = P * \sigma$, $P'' = P * \rho$. Then

$$\begin{aligned}\sup_{x\in K_1}|P_1(x)| &\le \sup_{x\in K_1}|P'(x)| + \sup_{x\in K_1}|P''(x)| \\ &\le C\sup_{x\in L}|P(x)| + \epsilon\|P\|_\infty.\end{aligned}$$

But we know that $\|P_1\|_\infty \le C_1 \sup_{x\in K_1}|P_1(x)|$. Altogether it implies

$$\|P_1\|_\infty \le C_1 C \sup_{x\in L}|P(x)| + \epsilon C_1\|P\|_\infty. \tag{7.4}$$

We have a similar estimate for P_2. Then

$$\|P\|_\infty \le \|P_1\|_\infty + \|P_2\|_\infty \le C_3 \sup_{x\in L}|P(x)| + \epsilon(C_1+C_2)\|P\|_\infty. \tag{7.5}$$

Finally (7.5) ends the proof if $\epsilon(C_1+C_2) < 1$.

We now consider the second assertion and assume $Q(\Lambda)$. Using Lemma 4.7 with $x_0 = m \in \mathbb{Z}$ we know that there exists a measure μ_m carried by K and such that $\|\mu_m\| \le C$ and $\widehat{\mu}_m(\lambda) = \exp(2\pi i m\lambda)$. Given a complex number z with $|z| = 1$ we select a sequence m_j such that $\exp(2\pi i m_j\alpha) \to z$, $j \to \infty$. Using a standard compactness argument we find a measure μ_z carried by K such that $\widehat{\mu}_z(k) = 1$, $k \in \Lambda_1$, while $\widehat{\mu}_z(\alpha k) = z^k, k \in \Lambda_2$. It then suffices to average with respect to z to obtain the required measure τ. The averaging procedure can be achieved in two steps. First we let $z_k = \exp(2\pi ik/N)$ be a Nth root of unity, we then average μ_{z_k} on $k \in \{0, \dots, N-1\}$ to obtain a measure ν_N and finally τ is a limit of a subsequence ν_{N_j}.

We now conclude the proof of Theorem 7.7. It uses the following lemma:

Lemma 7.9. *Let τ be a complex Radon measure with a finite total mass and let $\tau = \tau_0 + \rho$ the decomposition of τ into an atomic component τ_0 and a continuous measure ρ. If $\widehat{\tau} = 0$ on $\mathbb{N}$ then $\widehat{\tau}_0 = 0$ on $\mathbb{Z}$.*

Indeed we have $\widehat{\tau}_0(k) = r_k$, $k \in \mathbb{N}$, and $\lim_{N\to\infty} N^{-1}\left(\sum_1^N |r_k|^2\right) = 0$. But $\widehat{\tau}_0(k)$ is almost periodic on $\mathbb{Z}$. It implies $\widehat{\tau}_0(k) = 0$, $k \in \mathbb{Z}$, as announced.

To prove Theorem 7.7 we argue by contradiction and suppose that $Q(\Lambda)$ holds and $\alpha \notin \mathbb{Q}$. Then Theorem 7.7 implies the existence of a bounded Radon measure τ such that $\widehat{\tau}(k) = 0$, $k \in -\mathbb{N}$, and $\widehat{\tau}(\alpha k) = 1$, $k \in \mathbb{N}$. Lemma 7.9 yields $\widehat{\tau}_0(k) = 0$, $k \in \mathbb{Z}$ and $\widehat{\tau}_0(\alpha k) = 1$, $k \in \mathbb{Z}$. But this is impossible since $\widehat{\tau}_0(k)$ is uniformly continuous. This ends the proof of Theorem 7.7.

Here is an example where the first statement of Theorem 7.7 is used.

Theorem 7.10. *Let $\alpha \notin \mathbb{Q}$, $\epsilon > 0$, and let $\Lambda_{\alpha,\epsilon} = \mathbb{Z} \cup (\alpha\mathbb{Z} \setminus B_\epsilon)$ where B_ϵ is the set of all $\alpha k \in \alpha\mathbb{Z}$ whose distance to $\mathbb{Z}$ does not exceed ϵ. Then $Q(\Lambda_{\alpha,\epsilon})$ holds.*

We set $\Lambda_1 = \mathbb{Z}$ and $\Lambda_2 = \alpha\mathbb{Z} \setminus B_\epsilon$. We shall construct an atomic measure τ with a finite total mass such that $w = \widehat{\tau} = 1$ on Λ_1 and $w = 0$ on Λ_2. Let w be the $1-$periodic function whose restriction to $[-1/2, 1/2]$ is a $\mathcal{C}^\infty$ function supported by $[-\epsilon, \epsilon]$ which is equal to 1 on $[-\epsilon/2, \epsilon/2]$. Then $w = 1$ on $\mathbb{Z} + [-\epsilon/2, \epsilon/2]$ and $w = 0$ outside $\mathbb{Z} + [-\epsilon, \epsilon]$. Therefore $w = 1$ on Λ_1 and $w = 0$ on Λ_2. Finally the Fourier series expansion of w is absolutely convergent. We have $w = \widehat{\tau}$ where τ is an atomic measure with a finite mass which ends the proof.

8. Kahane's problem for L^p−norms

In his pioneering paper [3] Kahane proposed the following generalization of the property $Q(\Lambda)$. Given $p \in [1, \infty]$ and a uniformly discrete set $\Lambda \subset \mathbb{R}^n$, property $Q(\Lambda, p)$ holds if there exist a compact set K and a constant C such that for every trigonometric sum

$$f(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x) \tag{8.1}$$

whose spectrum is contained in Λ we have

$$\sup_{y \in \mathbb{R}^n} \left(\int_{K+y} |f(x)|^p \, dx \right)^{1/p} \leq C \left(\int_K |f(x)|^p \, dx \right)^{1/p}. \tag{8.2}$$

Then $Q(\Lambda, \infty)$ coincides with $Q(\Lambda)$. Property $Q(\Lambda, 2)$ holds if and only if Λ is uniformly discrete. More precisely let K be a ball of radius R. Then (8.2) is true when $p = 2$ and $R > R(\Lambda)$.

Theorem 8.1. *Let us assume $Q(\Lambda, 1)$. Then every Radon measure μ which is a tempered distribution and whose distributional Fourier transform is a measure supported by Λ is an almost periodic measure.*

The proof is not difficult. It depends on the following lemmas.

Lemma 8.2. *If $Q(\Lambda, 1)$ holds there exist a compact set K and a constant C such that for every almost periodic function f whose spectrum is contained in Λ we have*

$$\sup_{y \in \mathbb{R}^n} \int_{K+y} |f(x)| \, dx \leq C \int_K |f(x)| \, dx. \tag{8.3}$$

To prove (8.3) it suffices to observe that f is a uniform limit of trigonometric sums whose frequencies belong to Λ.

Lemma 8.3. *If Λ is uniformly discrete and if $\sigma = \sum_{\lambda \in \Lambda} c(\lambda) \delta_\lambda$ is a tempered distribution there exist an integer N and a constant C such that*

$$|c(\lambda)| \leq C(1 + |\lambda|)^N \quad (\forall \lambda \in \Lambda). \tag{8.4}$$

Lemma 8.3 follows from the definition of a tempered distribution. We now prove Theorem 8.1. Let ϕ be a non negative $\mathcal{C}^\infty$ function supported by the unit ball and such that $\int \phi \, dx = 1$. Let $\phi_m(x) = m^n \phi(mx)$, $m \in \mathbb{N}$. Then (8.4) implies that the Fourier transform of $f_m = \mu * \phi_m$ is an atomic measure carried by Λ with a finite total mass. Therefore f_m is an almost periodic function and Lemma 8.2 yields

$$\sup_{y \in \mathbb{R}^n} \int_{K+y} |f_m(x)| \, dx \leq C \int_K |f_m(x)| \, dx. \tag{8.5}$$

But the RHS of (8.5) does not exceed $|\mu|(K + B)$ where B is the unit ball. The measure μ is the weak limit of the sequence f_m. We have $\sup_{y \in \mathbb{R}^n} (\int_{K+y} |f_m(x)| \, dx) \leq C|\mu|(K+B)$. Therefore $\sup_{y \in \mathbb{R}^n} |\mu|(K+y) \leq C|\mu|(K+B)$ and μ is translation bounded.

Let g be a compactly supported continuous function. Since μ is translation bounded $F = \mu * g$ is uniformly continuous. The convolution products $F * \phi_m$ are almost periodic functions since their Fourier transforms are atomic measures with a finite total mass. Moreover the sequence $F * \phi_m$ converges uniformly to F. Finally F is an almost periodic function and μ is an almost periodic measure.

Corollary 8.4. *If $\Lambda_{\alpha,\beta}$ is defined by* (4.7) $Q(\Lambda_{\alpha,\beta}, 1)$ *does not hold.*

To establish this fact it suffices to replace Proposition 5.9 by Theorem 8.1 in the proof of Theorem 4.10.

Proposition 8.5. $Q(\Lambda, 1)$ *does not imply* $Q(\Lambda)$.

Here is a one dimensional counter example. One starts with $E = \{\, 3^j + 2^k \mid 0 \le k \le j,\ j, k \in \mathbb{N}\,\}$. Then E is a $\Lambda(2)$ set. It means that there exists a constant C such that $\|P\|_2 \le C\|P\|_1$ for every trigonometric sum $P(x) = \sum_{k \in E} c(k)\exp(2\pi i k x)$. Here $\|P\|_1 = \int_0^1 |P(x)|\,dx$ and $\|P\|_2 = (\int_0^1 |P(x)|^2\,dx)^{1/2}$. Let $r_{j,k}$ be a sequence of positive real numbers such that (a) $\sum_{0 \le k \le j} |r_{j,k}| = \beta < \frac{1}{4\pi}$ and (b) the collection $\{0\} \cup \{\, r_{j,k} \mid 0 \le k \le j,\ j, k \in \mathbb{N}\,\}$ is linearly independent over $\mathbb{Q}$. Then we have:

Theorem 8.6. *Let $\lambda_{j,k} = 3^j + 2^k + r_{j,k}$, $0 \le k \le j$, $j, k \in \mathbb{N}$. The set $\Lambda = \{\lambda_{j,k}\}$ satisfies $Q(\Lambda, 1)$ but not $Q(\Lambda)$.*

The following lemma will be used in the proof of Theorem 8.6

Lemma 8.7. *For every $1-$periodic continuous function $f(x) = \sum_{-\infty}^{\infty} c(k)\exp(2\pi i k x)$ and every summable sequence $r(k)$, $k \in \mathbb{Z}$, of complex numbers we have:*

$$\int_0^1 \left| \sum_{-\infty}^{\infty} r(k)c(k)\exp(2\pi i k x) \right| dx \le \sum_{-\infty}^{\infty} |r(m)| \int_0^1 \left| \sum_{-\infty}^{\infty} c(k)\exp(2\pi i k x) \right| dx.$$

This is obvious by the triangle inequality.

Let us prove the second statement in Theorem 8.6. The set E is not a Sidon set since it contains arbitrarily long direct sums $A + B$ where $\#A = \#B = N$. Therefore there exists a continuous $1-$periodic function $g_0(x) = \sum_{m \in E} c_m \exp(2\pi i m x)$ such that $\|g_0\|_\infty = 1$ and $\sum_{m \in E} |c_m| = \infty$. Let us write $\epsilon_m = r_{j,k}$ if $m = 3^j + 2^k, 0 \le k \le j$. We then consider

$$g(x) = \sum_{m \in E} c_m \exp(2\pi i (m + \epsilon_m)x). \tag{8.6}$$

We have $|g(x) - g_0(x)| \le 2\pi \sum_{m \in E} |c_m||\epsilon_m||x| \le C|x|$ and g is a continuous function. This function g is mean periodic, its Fourier transform is a sum of Dirac masses on Λ but g is not an almost periodic function since $\limsup_{x \to \infty} |g(x)| = \infty$ as Diophantine approximations show. Therefore $Q(\Lambda)$ does not hold.

Let us prove $Q(\Lambda, 1)$. Let $f(x) = \sum_{m \in E} a_m \exp(2\pi i (m + \epsilon_m)x)$ be an arbitrary trigonometric sum whose frequencies belong to Λ. We expand

$$\exp(2\pi i \epsilon_m x) = \sum_0^{\infty} \frac{(2\pi i \epsilon_m x)^k}{k!}$$

which leads to

$$f(x) = \sum_0^\infty f_k(x).$$

But Lemma 8.7 yields

$$\int_0^1 |f_k(x)|\, dx \le \frac{(2\pi\beta)^k}{k!} \int_0^1 |f_0(x)|\, dx.$$

Therefore $\pi\beta < 1/4$ implies

$$\int_0^1 |f(x)|\, dx \ge \Big(1 - \frac{2\pi\beta}{1-2\pi\beta}\Big) \int_0^1 |f_0(x)|\, dx.$$

Finally we use the classical fact that E is a $\Lambda(2)$ set and obtain

$$\int_0^1 |f(x)|\, dx \simeq \Big(\int_0^1 |f_0(x)|^2\, dx\Big)^{1/2}.$$

But this L^2-norm is $\big(\sum_{m\in E}|a_m|^2\big)^{1/2}$. Everything being translation invariant we have also $\int_q^{q+1}|f(x)|\, dx \simeq \big(\sum_{m\in E}|a_m|^2\big)^{1/2}$. Finally (8.1) holds with $p = 1$ and $K = [0, 1]$ which ends the proof of Theorem 8.6.

9. Growth estimates

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. Let us assume property $T(\Lambda)$ (Definition 4.1). Let $\mathcal{C}_\Lambda$ denote the topological vector space consisting of all mean periodic functions whose spectrum is simple and contained in Λ. If Kahane's property $Q(\Lambda)$ is not satisfied there exists a function $f \in \mathcal{C}_\Lambda$ which is not bounded. How large can f be at infinity? To answer this issue let us return to the definition of mild sets which was already given in Section 4, Definition 4.8.

Definition 9.1. *A mild weight function $\omega(x)$, $x \in \mathbb{R}^n$, satisfies the three following properties*

- *(a) ω is a continuous function and $\omega(x) \ge 1$*
- *(b) $\omega(x+y) \le \omega(x)\omega(y)$, $x, y \in \mathbb{R}^n$*
- *(c) There exist a constant C and an exponent m such that*

$$\omega(x) \le C(1+|x|)^m,\ x \in \mathbb{R}^n.$$

We now investigate the behavior at infinity of mean periodic functions $f \in \mathcal{C}_\Lambda$.

Definition 9.2. *A closed and discrete set Λ of real numbers is a* mild *set of frequencies if there exist a mild weight function ω and a compact set K such that every $f \in \mathcal{C}_\Lambda$ satisfies*

$$\forall x \in \mathbb{R}, \quad |f(x)| \le \omega(x) \sup_{y\in K}|f(y)|. \tag{9.1}$$

When (9.1) is not satisfied there are two options:

- (i) Either there exist a "large weight" ω and a compact set K for which (9.1) holds
- (ii) Or (9.1) never holds whatever be the weight ω and the compact K.

If the second option occurs we say that Λ is *wild*. An example is given now. Let $\theta > 2$ be a real number. We define Λ_θ as the set of all finite sums $\sum_{k\geq 0} \epsilon_k \theta^k$, $\epsilon_k \in \{0,1\}$. Let us assume that θ is not a Pisot–Thue–Vijayaraghavan number. We consider the sequence $P_m(x)$ of finite products $P_m(x) = \prod_0^{m-1}(\frac{1+\exp(2\pi i\theta^k x)}{2})$. The spectrum of P_m is contained in Λ. By Pisot's theorem we know that $|P_m(x)| = \prod_0^{m-1}|\cos(\pi\theta^k x)|$ converge uniformly to 0 on every compact set not containing the origin. We have $P_m(0) = 1$. Assuming by contradiction that (9.1) is true for some weight ω, we choose $x_0 \notin K$ and consider $R_m(x) = P_m(x - x_0)$. Then R_m converges uniformly to 0 on K while $R_m(x_0) = 1$. We have reached the required contradiction.

The following theorem gives a sufficient condition implying that a closed and discrete set Λ is a mild set of frequencies.

Theorem 9.3. *Let $M \subset \mathbb{R}^n$ be a uniformly discrete set and let $\sigma_M = \sum_{m\in M} c(m)\delta_m$ be an atomic measure supported by M. Let us assume that the distributional Fourier transform $\widehat{\sigma}_M$ of σ_M is a Radon measure. Let μ be the inverse Fourier transform of σ_M and let us define $w(x)$ by $w(x) = \int_{|y-x|\leq 1} d|\mu|(y)$. Let us assume that there exists a mild weight $\omega(x) \geq w(x)$. Let $\Lambda = \{\, m \in M \mid c(m) = 1\,\}$.*

Then there exist a compact set K and a constant C such that every $f \in \mathcal{C}_\Lambda$ satisfies

$$|f(x)| \leq C\omega(x) \sup_{y\in K}|f(y)|. \tag{9.2}$$

For proving this theorem we define $\beta > 0$ by

$$\inf\{\, |m - m'| \mid m \neq m',\, m, m' \in M\,\} = \beta > 0.$$

Let $0 < r < r' < \beta/2$, let B_r (resp. B'_r) the ball centered at 0 with radius r (resp. r'). Let ϕ be a function in the Schwartz class $\mathcal{S}$ such that $\widehat{\phi} = 1$ on B_r and $\widehat{\phi} = 0$ outside B'_r. Let μ_y be the Radon measure μ translated by $-y$ and let $\chi_y(x) = \exp(2\pi i xy)$. Then the Fourier transform of the product $\phi\mu_y$ is the convolution product $\widehat{\phi} * \chi_y\sigma_M$. Let $E = \Lambda + B_r$. The following lemma resumes this discussion:

Lemma 9.4. *We have*

$$\widehat{\phi\mu_y}(\xi) = \sum_{\lambda\in\Lambda} \exp(2\pi i\lambda\cdot y)\,\widehat{\phi}(\xi - \lambda) + R(\xi) \tag{9.3}$$

where the function R vanishes on E.

Therefore $\widehat{\phi\mu_y}(\lambda + s) = \exp(2\pi i\lambda \cdot y)$ if $\lambda \in \Lambda$, $s \in B_r$. Let PW_E^∞ denote the Paley–Wiener space consisting of all functions $f \in L^\infty$ whose Fourier transform is supported by E [7]. Then every $f \in PW_E^\infty$ can be written as $f(x) = \sum_{\lambda\in\Lambda} \exp(2\pi i\lambda x) g_\lambda(x)$ where $g_\lambda \in PW_{B_r}^\infty$.

Definition 9.5. *For $y \in \mathbb{R}^n$ the twisted translation operator $T_y\colon PW_E^\infty \to PW_E^\infty$ is defined by $T_y[\sum_{\lambda\in\Lambda} \exp(2\pi i\lambda x)g_\lambda(x)] = \sum_{\lambda\in\Lambda} \exp(2\pi i\lambda(x+y))g_\lambda(x)$.*

Then we have

Lemma 9.6. *For $f \in PW_E^\infty$ we have $T_y(f) = (\phi\mu_y) * f$.*

Indeed Lemma 9.6 immediately follows from Lemma 9.4.

Lemma 9.7. *There exists a constant C such that the total mass of the Radon measure $\phi\mu_y$ does not exceed $C\omega(y)$.*

This is immediately implied by the three properties of a mild weight. We conclude:

Lemma 9.8. *The operator norm of $T_y\colon PW_E^\infty \to PW_E^\infty$ does not exceed $C\omega(y)$.*

We are now ready to conclude the proof of Theorem 9.3. It suffices to apply Lemma 9.8 to a product $f = Pg$ where $P(x) = \sum_{\lambda\in\Lambda} c_\lambda \exp(2\pi i\lambda x)$ is an arbitrary trigonometric polynomial whose spectrum is contained in Λ and g is a fixed function in the Schwartz class whose Fourier transform is supported by B_r and such that $g(0) = 1$. Then $T_y(Pg) = P(\,\cdot + y)g(\,\cdot\,)$ and Lemma 9.8 yields $\|P(\,\cdot + y)g(\cdot)\|_\infty \leq C\omega(y)\|Pg\|_\infty$. It implies

$$|P(y)| \leq C\omega(y)\|Pg\|_\infty. \tag{9.4}$$

Let us consider $\eta(y) = \frac{|P(y)|}{\omega(y)}$ and rewrite (9.4) as

$$\eta(y) \leq C_0 \sup_{|x|\leq T} |P(x)| + C \sup_{|x|\geq T} |\eta(x)g(x)\omega(x)|. \tag{9.5}$$

We choose T large enough to ensure $C|g(x)\omega(x)| \leq 1/2$ when $|x| \geq T$. This is made possible since ω has a polynomial growth at infinity while g is rapidly decaying. Taking the supremum of the LHS of (9.5) with respect to y one obtains $\|\eta\|_\infty \leq C \sup_{|x|\leq T}|P(x)| + (1/2)\|\eta\|_\infty$ which ends the proof.

Theorem 9.9. *If $\Lambda = \Lambda_{\alpha,\beta}$ is defined by (4.7) every mean periodic function $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ is $O(\sqrt{|x|})$ at infinity. Moreover this estimate is optimal.*

To prove Theorem 9.9 it suffices to estimate $w(x) = \int_x^{x+1} d|\mu|$ when μ is the measure defined by (6.5) in Proposition 6.2. We have

$$\gamma(p, q+\alpha p) = \int_{\mathbb{T}} \exp\bigl[-2\pi i\bigl(pt + (q+\alpha p)\theta(t)\bigr)\bigr]\, dt$$

where $\theta(t) = \beta\sin t$. If $x \leq q+\alpha p < x+1$ then $q+\alpha p = x+s$, $0\leq s<1$. It leads to

$$\exp\bigl[-2\pi i\bigl(pt + (q+\alpha p)\theta(t)\bigr)\bigr] = \sum_0^\infty \frac{(-2\pi i s\,\theta(t))^k}{k\,!} \exp\bigl[-2\pi i\bigl(pt + x\theta(t)\bigr)\bigr].$$

Let us observe that given p and x there exists only one q such that $x \leq q+\alpha p < x+1$. Finally

$$\begin{aligned}\int_x^{x+1} d|\mu| &= \sum_{x\leq q+\alpha p<x+1} |\gamma(p, q+\alpha p)| \\ &\leq \sum_{k=0}^\infty \frac{(2\pi)^k}{k\,!} \sum_{p\in\mathbb{Z}} \left| \int_{\mathbb{T}} \theta(t)^k \exp\bigl[-2\pi i\bigl(pt + x\theta(t)\bigr)\bigr]\, dt \right|.\end{aligned}$$

Using the definition of the norm in the Wiener algebra $A = A(\mathbb{T})$ we obtain

$$\sum_{p\in\mathbb{Z}}\left|\int_{\mathbb{T}} \exp(-2\pi ipt)\,\theta(t)^k \exp[-2\pi ix\theta(t)]\,dt\right| \le \|\theta^k\|_A\,\|\exp[2\pi ix\theta\|_A.$$

But $\theta(t) = \sin(2\pi t)$ implies

$$\|\exp[2\pi ix\theta\|_A \le C\sqrt{1+|x|}$$

which ends the proof. We now prove the optimality. Let W a weight function and K a compact interval such that every $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ satisfies

$$|f(x)| \le W(x)\sup_{y\in K}|f(y)|. \tag{9.6}$$

We want to prove that $W(x) \ge C\sqrt{x}$. Let μ be the inverse Fourier transform of σ_Λ. Let g be an arbitrary continuous function supported by $[-1,1]$ and normalized by $\|g\|_\infty \le 1$. Then $f = \mu * g$ belongs to $\mathcal{C}_{\Lambda_{\alpha,\beta}}$. Therefore if $L = K + [-1,1]$ we have

$$|\mu * g(x)| \le W(x)\int_L d|\mu|. \tag{9.7}$$

Taking the supremum over g yields

$$\int_{x-1}^{x+1} d|\mu| \le C\,W(x) \tag{9.8}$$

with $C = \int_L d|\mu|$. But the LHS of (9.8) is larger than $c\sqrt{1+|x|}$, $c > 0$, which ends the proof.

The uniformly discrete set Λ_α defined in Theorem 7.7 is mild. To prove this remark it suffices to observe that every $f \in \mathcal{C}_{\Lambda_\alpha}$ satisfies the convolution equation

$$f(x) - f(x-1) - f(x-\alpha) + f(x-1-\alpha) = 0. \tag{9.9}$$

It implies that $u(x) = f(x) - f(x-\alpha)$ is $1-$periodic. Therefore u is uniformly bounded. We have $|f(x) - f(x-\alpha)| \le C$ which implies $f(x) = O(|x|)$ at infinity.

This estimate is optimal if $f \in \mathcal{C}_{\Lambda_\alpha}$. Let us begin with an easy observation.

Lemma 9.10. *The estimate $f(x) = O(|x|)$ at infinity is optimal for solutions of* (9.9).

As we already observed every $f \in \mathcal{C}_{\Lambda_\alpha}$ is a solution of (9.9) but the converse is not true. We now prove Lemma 9.10. There exist two sequences of integers m_j, $j \in \mathbb{N}$, and $q_j \in \mathbb{N}$ such that $m_j\alpha - q_j \to 0$, $j \to \infty$. We then consider $f_j(x) = \frac{\exp(2\pi i m_j\alpha x) - \exp(2\pi i q_j x)}{m_j\alpha - q_j}$. Then we have $|f_j(x)| \le 2\pi|x|$ everywhere and $|f_j(x)| \to 2\pi|x|$, $j \to \infty$ which proves optimality in the general context of solutions to (9.9).

We now prove the optimality when $f \in \mathcal{C}_{\Lambda_\alpha}$. Let ω be a weight function and K be a compact set such that every $f \in \mathcal{C}_{\Lambda_\alpha}$ satisfies $|f(x)| \le \omega(x)\sup_{y\in K}|f(y)|$. Then for every integer $m \in \mathbb{N}$ there exists a Radon measure μ_m supported by K and such that $\|\mu_m\| \le \omega(m)$ and $\widehat{\mu_m}(k) = 1$, $\widehat{\mu_m}(\alpha k) = \exp(2\pi ikm\alpha)$, $k \in \mathbb{N}$. As we did in the proof of Theorem 7.7 we decompose μ_m into a continuous part ν_m and an atomic part τ_m. We have as above $\widehat{\tau_m}(k) = 1$, $\widehat{\tau_m}(\alpha k) = \exp(2\pi ikm\alpha)$, $k \in \mathbb{Z}$. Since τ_m is supported by

K and satisfies $\|\tau_m\| \leq \|\mu_m\|$ it implies that the weight ω also governs the behavior at infinity of all solutions f to (9.9). Therefore $\omega(x) \geq |x|$ as announced.

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CMLA, ENS-Cachan, CNRS, Université Paris-Saclay, France
Email address: yves.meyer305@orange.fr

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