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MEAN-PERIODIC FUNCTIONS AND IRREGULAR SAMPLING

YVES MEYER

ABSTRACT. The theory of irregular sampling raises many mathematical problems. A. Olevskii and A. Ulanovskii proved that *universal sampling sets* exist. This remarkable discovery is given a new proof here. A problem raised sixty years ago by Jean-Pierre Kahane on mean-periodic functions is paving the way to our surprising approach.

1. LOST OPPORTUNITIES

A few months ago I was preparing the 2018 Onsager lecture I had to deliver in Trondheim. I checked the bibliography. Most of the time I forget to do it. I used Google and entered the key words of my talk. Google pointed at a paper by Jean-Pierre Kahane entitled *Sur les fonctions moyenne-périodiques bornées*. It was published in 1957 in Annales de l'Institut Fourier [5].

To my great shame I had not read this paper until now. It is so surprising since Kahane was my colleague at the mathematical department of Université Paris-Sud (Orsay) from 1967 to 1981 and we had many mathematical discussions. We also had some painful political disputes. The issue discussed by Kahane in his remarkable 1957 paper is a particular instance of a broader problem: Can we infer the spectral properties of a discrete point set $\Lambda \subset \mathbb{R}^n$ from its arithmetical structure?

More precisely Kahane proposed to investigate the structure of the discrete sets $\Lambda \subset \mathbb{R}^n$ which have the following property: *Every mean-periodic function whose spectrum is contained in Λ is necessarily an almost-periodic function.* This terminology will be defined in Sections 3 and 4.

In 1968, eleven years after the publication of Kahane's paper, I attacked exactly the same problem. It paved the way to the construction of *model sets* (model sets will be defined in Section 15). I described the spectral properties of these *model sets*. I wrote a book [11] on these issues. It was published by North-Holland in 1972. Today *model sets* are used to model quasi-crystals.

In 1974 Roger Penrose constructed his famous pavings. De Bruijn (March 20, 1980) elucidated the arithmetical structure of the set V of vertices of a particular Penrose paving. Using today's terminology V is a *model set*. Therefore this set V is a solution to the problem raised by Kahane.

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De Bruijn was unaware of Kahane's problem. He was also unaware of my book where the theory of *model sets* was detailed.

In 1985 the physicist Jean Lascoux (the same Lascoux who played such a seminal role in the wavelet saga) asked whether my previous work in number theory was related to Penrose pavings. I did not recognize my *model sets* and I said "no". I was so dumb! Fortunately Robert Moody understood the connection between De Bruijn's interpretation of Penrose pavings and my previous work.

On April 8, 1982, Daniel Shechtman elaborated some chemical alloys presenting a five-fold symmetry in their diffraction pattern. A mathematical theory explaining such patterns was elaborated by M. Duneau, D. Gratias, and A. Katz. Duneau, Gratias, and Katz rediscovered my *model sets*.

In 2010 Basarab Matei and I proved that *simple quasi-crystals are universal sampling sets* (this terminology will be explained later on). This gave a simpler solution to a problem raised and solved in 2006 by A. Olevskii and A. Ulanovskii. Our solution looked quite original. Today I am ashamed. Let me explain why I am so embarrassed.

Two months ago I dove into the book by A. Olevskii and A. Ulanovskii entitled *Functions with Disconnected Spectrum* [12]. Huge surprise: Using a result from this book I understood that my joint work with Matei was not original. It is an obvious corollary of a theorem which was proved forty five years ago [10]. This unexpected approach to irregular sampling will be detailed in this contribution (Section 15). Kahane's problem is deeply rooted in this new proof as it will be proved below.

The achievements of Kahane, Penrose, De Bruijn, Moody, Shechtman, Olevskii, Ulanovskii, and my contributions fit together like pieces of a jigsaw puzzle whose global structure was not immediately understood. Pursuing distinct goals we were studying the same problem at the same time. Unfortunately we did not collaborate. We were genuinely obeying our inner voices. These inner voices were communicating in a mysterious way. How is it possible? In his Nobel lecture, Joseph Brodsky addresses this issue. He explains how artists obeying their inner voice are indeed collaborating without even noticing it:

> *Art is a recoilless weapon, and its development is determined not by the individuality of the artist, but by the dynamics and the logic of the material itself, by the previous fate of the means that each time demand (or suggest) a qualitatively new aesthetic solution.*

Mathematicians are players of an orchestra directed by a "hidden conductor". As Brodsky is suggesting this "hidden conductor" is the dynamics of mathematics. This is true for the whole of Science. The pioneers who launched the *Institute for the Unity of Science* in 1947 predicted that the hierarchy between distinct scientific fields advocated by Auguste Comte would disappear in modern Science.

In his essay on the *Institute for the Unity of Science* [3], Peter Galison described the revolutionary goals of the Institute:

> *This Comtian hierarchy is replaced by the orchestration of different instruments, each distinct but brought together to accomplish something bigger than any could do individually.*

Jean Delsarte, Laurent Schwartz, Jean-Pierre Kahane, Roger Penrose, Nicolaas De Bruijn, Daniel Shechtman, Robert Moody, Alexander Olevskii, and Alexander Ulanovskii were playing a remarkable music I had the chance to hear.

2. Organization

(3) Almost-periodic functions (Harald Bohr)
(4) Mean-periodic functions (Jean Delsarte and Laurent Schwartz)
(5) Kahane's problem: almost-periodic versus mean-periodic
(6) Pisot–Thue– Jijayaraghavan numbers
(7) Local structure of almost-periodic functions

(8) Paley-Wiener spaces
(9) Generalized Shannon's theorem
(10) Sparsity
(11) Stable sampling
(12) Stable interpolation
(13) Landau's theorem
(14) Universal sampling sets
(15) Sampling on quasi-crystals (the new proof).

Sections 3 to 14 do not contain much original results. They are providing many definitions and some heuristics which pave the way to the new proof of the main theorem. This proof is detailed in Section 15 which is self-contained.

3. Almost-periodic functions

The function $f(x) = \cos(x) + \cos(2x/3)$ of the real variable x is periodic with period 6π : $f(x + 6\pi) = f(x)$. The set of periods of f is $6\pi\mathbb{Z}$. Each interval of length 6π contains a period of f. These trivial observations pave the way to the theory of almost-periodic functions. The function $g(x) = \cos(x)+\cos(\sqrt{2}x)$ is not a periodic function. It is our first example of an almost-periodic function. Let us compare $g(x)$ to the translated functions $g(x + 2k\pi) = \cos(x) + \cos(\sqrt{2}(x + 2k\pi)),\ k \in \mathbb{Z}$. Diophantine approximation theory implies the following fact: There exists a constant C such that for every $\epsilon > 0$ each interval I of length C/ϵ contains an integer k_I such that $|\sqrt{2}k_I - m_I| \le \epsilon,\ m_I \in \mathbb{Z}$. Then

$$|g(x + 2k_I\pi) - g(x)| \le 2\pi\epsilon.$$

The function g repeats itself with an arbitrarily small error after a period of time given by $2k_I\pi$. We have just defined an almost-periodic function.

We now consider functions of n real variables. We follow Harald Bohr and define almost-periodic functions in full generality. The sup norm of a continuous bounded function f is defined by $\|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|$.

Definition 3.1. *Let f be continuous and bounded on $\mathbb{R}^n$. For $\tau \in \mathbb{R}^n$ let f_τ be the translated function defined by $f_\tau(x) = f(x - \tau)$. Let $\epsilon \in (0, 2)$. Then $\tau \in \mathbb{R}^n$ is an ϵ almost-period of f if*

$$\|f_\tau - f\|_\infty \le \epsilon\|f\|_\infty \tag{1}$$

A set of points $\Lambda \subset \mathbb{R}^n$ is relatively dense if there exists a constant C such that for every $x \in \mathbb{R}^n$ the ball $B(x, C)$ centered at x with radius C contains at least one $y \in \Lambda$. This definition is due to Besicovitch. In other words Λ is relatively dense if there exists a compact set $K \subset \mathbb{R}^n$ such that $\Lambda + K = \mathbb{R}^n$.

Definition 3.2. *A continuous function f on $\mathbb{R}^n$ is almost-periodic in the sense of Harald Bohr if it is bounded and if for every $\epsilon \in (0, 2)$ the set Λ_ϵ of ϵ almost-periods τ of f is relatively dense.*

An almost-periodic function is uniformly continuous. The space $\mathcal{B}$ of almost-periodic functions is a Banach space when it is equipped with the sup norm $\|f\|_\infty$.

Definition 3.3. *A trigonometric polynomial is a finite sum*

$$P(x) = \sum_{\omega\in F} c(\omega) \exp(2\pi i\omega \cdot x)$$

where the coefficients $c(\omega)$ are arbitrary complex numbers.

A trigonometric polynomial is an almost-periodic function. If f in an almost-periodic function there exists a sequence P_j of trigonometric polynomials such that: $\|f - P_j\|_\infty \to 0,\ j \to \infty$.

Let f be an almost-periodic function. Let c_n be the inverse of the volume of the unit ball. Then the mean value of f exists, is defined by $\mathcal{M}(f) = \lim_{R\to\infty} c_n R^{-n} \int_{B(x,R)} f(x)\,dx$ and is attained uniformly in x. For each $\omega \in \mathbb{R}^n$ the product $\exp(-2\pi i\omega \cdot x)\, f(x)$ is also an almost-periodic function. This leads to the following definition.

Definition 3.4. *Let f be an amost-periodic function and $\omega \in \mathbb{R}^n$. We define the Fourier coefficients of f by $\widehat{f}(\omega) = \mathcal{M}(\overline{\chi}_\omega f)$ where $\chi_\omega(x) = \exp(2\pi i\omega \cdot x)$ and $\overline{\chi_\omega(x)}$ is the complex conjugate of $\chi_\omega(x)$.*

The notation $\widehat{f}(\omega)$ can be confusing since $\widehat{f}(\omega)$ is not the "value" at ω of the distributional Fourier transform $\widehat{f}$ of f. If f is almost-periodic, so is $|f|^2$, and one has $\mathcal{M}(|f|^2) = \sum_{\omega\in\mathbb{R}^n} |\widehat{f}(\omega)|^2$. Therefore the set S of frequencies ω for which $\widehat{f}(\omega) \neq 0$ is at most a numerable set.

Definition 3.5. *The spectrum $\sigma(f)$ of f is the set $S = \{\omega \in \mathbb{R}^n;\ \widehat{f}(\omega) \neq 0\}$.*

The distributional spectrum of a tempered distribution f is the closed support of its Fourier transform $\widehat{f}$. How is the distributional spectrum of an almost-periodic function f related to $\sigma(f)$?

Lemma 3.1. *If f is an almost-periodic function, its distributional spectrum is the closure of $\sigma(f)$.*

An example is the almost-periodic function $f(x) = \sum_0^\infty 2^{-k} \cos(2\pi 2^{-k} x)$. Then $\sigma(f) = \{\pm 2^{-k}, k = 0, 1, \ldots\}$ while the distributional spectrum of f is $\sigma(f) \cup \{0\}$. Any numerable set E is the spectrum of an almost-periodic function.

The following definition plays a key role in a problem raised by Jean-Pierre Kahane and studied in Section 5.

Definition 3.6. *Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary set. Then $\mathcal{B}_\Lambda$ denotes the vector space consisting of all almost-periodic functions f whose spectrum $\sigma(f)$ is contained in Λ.*

Here are some examples. If $\Lambda = \mathbb{Z}$ then $\mathcal{B}_\Lambda$ is the vector space of all 1-periodic continuous functions. Let $\alpha \notin \mathbb{Q}$ and let $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$. Then we have $f \in \mathcal{B}_\Lambda$ if and only if $f = u + v$ where u is a 1-periodic continuous function and v is a $1/\alpha-$periodic continuous function. If $\Lambda = \{2^{-k}, k = 0, 1, \ldots\}$ then $f \in \mathcal{B}_\Lambda$ if and only if the Fourier expansion of f is absolutely convergent: $f(x) = \sum_0^\infty c_k \exp(2\pi i 2^{-k} x)$ where $\sum_0^\infty |c_k| < \infty$.

The subspace $\mathcal{V}_\Lambda \subset \mathcal{B}_\Lambda$ consists of all finite trigonometric sums $\sum_{\lambda\in\Lambda} c(\lambda) \exp(2\pi i\lambda \cdot x)$ whose frequencies belong to Λ. Then $\mathcal{V}_\Lambda$ is dense in $\mathcal{B}_\Lambda$.

4. Mean-periodic functions

The theory of mean-periodic functions was elaborated in the fifties by Jean Delsarte, Leon Ehrenpreis, Bernard Malgrange, and Laurent Schwartz.

Let us begin with the simplest example. The Fibonacci sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_0 = 0,\ x_1 = 1,\ x_{n+1} = x_n + x_{n-1},\ n \in \mathbb{N}$, is a mean-periodic sequence. In general a mean-periodic sequence $x = (x_n)_{n\in\mathbb{Z}}$ of real or complex numbers is a solution of a recurrence relation $\sum_0^N c(k) x_{n-k} = 0$. This equation can be written $x * \mu = 0$ where μ is a finite sum of weighted Dirac measures.

We now move from the discrete to the continuous world. The space $\mathcal{B}$ of almost-periodic functions is a Banach space when the topology is defined by the sup norm on $\mathbb{R}^n$. What will happen in a billion of years is as relevant as what is happening now. In the case of mean-periodic functions the present is more important than the future. Therefore mean-periodic functions are antagonist to almost-periodic functions. That remark stresses the relevance of Kahane's problem (Section 5).

Let $\mathcal{C}(\mathbb{R}^n)$ denote the topological vector space of all continuous functions on $\mathbb{R}^n$. No growth condition at infinity is required. The vector space $\mathcal{C}(\mathbb{R}^n)$ is equipped with the topology $\mathcal{T}$ of uniform convergence on compact sets. This topology is much weaker than the one used in the theory of almost-periodic functions.

Instead of developing a general theory of mean-periodic functions let us focus on a relevant example. For the sake of simplicity this example is studied in one dimension. Let $\alpha \notin \mathbb{Q}$, $\alpha > 0$, and consider the functional equation

$$f(x-\alpha-1) - f(x-\alpha) - f(x-1) + f(x) = 0 \tag{2}$$

where f is a continuous function of the real variable x. Let $\mu_\alpha = \delta_\alpha - \delta_0$ where $\delta_\alpha = \delta(x-\alpha)$ is the Dirac measure at α. Then $\mu_1 = \delta_1 - \delta_0$ and (2) can be rewritten as $f * \mu_\alpha * \mu_1 = 0$. Obvious solutions of (2) are continuous solutions of $f * \mu_\alpha = 0$ or of $f * \mu_1 = 0$. We find periodic functions of period α in the first case and of period 1 in the second. A sum $f = u + v$ between two such periodic continuous functions is still a solution of (2). *Are there other continuous solutions of* (2)? This is the fundamental problem of the theory of mean-periodic functions. If one imposes $f \in \mathcal{B}$ the answer is no. *The only almost-periodic solutions of* (2) *are the trivial solutions* $f = u + v$. But non trivial continuous solutions $f \notin \mathcal{B}$ exist. Here is the construction. By Hurwitz's theorem there are infinitely many rational numbers p_j/q_j, $p_j, q_j \in \mathbb{N}$, $j \geq 1$, such that $|\frac{1}{\alpha} - \frac{p_j}{q_j}| < \frac{1}{q_j^2\sqrt{5}}$. We can assume $q_{j+1} \geq 2q_j$. We define a continuous function of the real variable x by

$$f(x) = \sum_1^\infty [\exp(2\pi i q_j x/\alpha) - \exp(2\pi i p_j x)] \tag{3}$$

Since $|\exp(2\pi i q_j x/\alpha) - \exp(2\pi i p_j x)| \leq C|x|/q_j$ and since the q_j have an exponential growth the series (3) converges uniformly on compact sets to a continuous function $f(x)$ which satisfies (2).

Proposition 4.1. *The function f defined by* (3) *does not belong to L^∞.*

Here is the proof. Let $f_m = \sum_1^m [\exp(2\pi i q_j x/\alpha) - \exp(2\pi i p_j x)]$ be the partial sums of (3). If $f \in L^\infty$ it will be shown below that $\|f_m\|_\infty \leq C$. This is wrong: $\|f_m\|_\infty = 2m$.

Here are the details of this argument. Let ϕ_j be a test function whose Fourier transform is 1 on $[-q_j - 1, q_j + 1]$ and vanishes outside $[-q_{j+1} - 1, q_{j+1} + 1]$. Since $q_{j+1} \geq 2q_j$ we can also impose $\|\phi_j\|_1 \leq C$. Let us argue by contradiction and assume that f belongs to L^∞. We then have $\|f * \phi_j\|_\infty \leq C$.

Lemma 4.1. $f * \phi_j = f_j$.

If Lemma 4.1 is granted, Proposition 4.1 is immediate. Indeed $f \in L^\infty$ implies $\|f_j\|_\infty \leq C$. But $\|f_j\|_\infty = 2j$ as Diophantine approximations show.

We return to the proof of Lemma 4.1. We have $|f_m| \leq C|x|$ and f_m tends to f in a weighted L^∞ space. More precisely the L^∞ norm of $\frac{|f_m(x) - f(x)|}{1+|x|}$ tends to 0 as m tends to ∞. It implies $f * \phi_j = \lim_{m\to\infty} f_m * \phi_j$ for every j and the convergence holds in the same weighted L^∞ space. But by the definition of ϕ_j we have $f_m * \phi_j = f_j$ if $m \geq j$ which ends the proof of Lemma 4.1.

A continuous solution of (2) is a mean-periodic function. The general theory can now be developed. Two options exist. One can define a mean periodic function as a continuous solution of a convolution equation $f * \mu = 0$ where μ is a compactly supported complex Radon measure. Such an equation generalizes (2). A second option is to start with a spectrum Λ. In our previous example $\Lambda = \mathbb{Z} \cup \alpha^{-1}\mathbb{Z}$. Then any continuous solution f of (2) is the limit with respect to the topology of uniform convergence on compact sets of a sequence P_j of trigonometric polynomials whose frequencies belong to $\mathbb{Z} \cup \alpha^{-1}\mathbb{Z}$.

To define mean-periodic functions in full generality we replace $\mathbb{Z} \cup \alpha^{-1}\mathbb{Z}$ by an arbitrary closed and discrete set Λ. A set of points $\Lambda \subset \mathbb{R}^n$ is discrete if each $\lambda \in \Lambda$ is isolated in Λ. As above $\mathcal{V}_\Lambda$ is the vector space of all finite trigonometric sums $\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i \lambda \cdot x)$ whose frequencies λ belong to Λ.

Definition 4.1. *With these notations $\mathcal{C}_\Lambda$ denotes the closure of the space $\mathcal{V}_\Lambda$ with respect to the topology $\mathcal{T}$ of uniform convergence on compact sets.*

The space $\mathcal{B}_\Lambda$ is defined by Definition 3.6. We obviously have $\mathcal{B}_\Lambda \subset \mathcal{C}_\Lambda$. Two possibilities can occur. Either $\mathcal{C}_\Lambda = \mathcal{C}(\mathbb{R}^n)$ or $\mathcal{C}_\Lambda \neq \mathcal{C}(\mathbb{R}^n)$. In the first case the space $\mathcal{C}_\Lambda$ is uninteresting. In the second case the Hahn Banach theorem implies the following: there exists a non trivial compactly supported complex Radon measure μ such that every $f \in \mathcal{C}_\Lambda$ satisfies $\mu * f = 0$. Then f is called a *mean-periodic function* (Definition 4.2 below) and we then say that the spectrum of f is contained in Λ.

Definition 4.2. *A function $f \in \mathcal{C}(\mathbb{R}^n)$ is mean-periodic if there exists a compactly supported complex Radon measure μ, $\mu \neq 0$, such that $f * \mu = 0$.*

This definition was proposed by Jean Delsarte. Such a convolution equation generalizes the recurrence relation satisfied by the Fibonacci sequence.

If Λ is uniformly discrete we have $\mathcal{C}_\Lambda \neq \mathcal{C}(\mathbb{R}^n)$. Let us prove this observation.

Definition 4.3. *A set of points $\Lambda \subset \mathbb{R}^n$ is uniformly discrete if there exists a $\beta > 0$ such that $|\lambda - \lambda'| > \beta > 0$ when $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Lambda$.*

Proposition 4.2. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. There exists a function g_Λ with the following properties (a) $g_\Lambda \neq 0$, (b) g_Λ is supported by a compact set $B \subset \mathbb{R}^n$, (c) $g_\Lambda \in L^2(B)$, and (d) the Fourier transform of g_Λ vanishes on Λ.*

Indeed for $a \notin \Lambda$, the set $\Lambda \cup \{a\}$ is a set of stable interpolation for PW^2_B whenever B is a ball and the diameter of B is large enough. This is a standard result on stable interpolation [5]. Stable interpolation will be studied below. Therefore there exists a function $g_\Lambda \in L^2(B)$, supported by B, such that $\widehat{g}_\Lambda = 0$ on Λ and $\widehat{g}_\Lambda(a) = 1$. Finally every $f \in \mathcal{C}_\Lambda$ satisfies $f * g_\Lambda = 0$. Therefore every $f \in \mathcal{C}_\Lambda$ is a mean-periodic function.

Here is an example of an almost-periodic function which are is mean-periodic. Let $g(x) = \sum_0^\infty 2^{-k} \sin(2\pi 2^{-k} x)$. Then g is obviously an almost-periodic function but is not a mean-periodic function. If μ is a compactly supported complex Radon measure $\mu * g = 0$ implies $\mu = 0$. Let us end this section with an example of a function f which is given by a Fourier series expansion but is neither a mean-periodic function nor an almost-periodic function. It is given by the following expansion $f(x) = \sum_0^\infty \sin(2\pi 2^{-k} x)$. This series is uniformly convergent on compact sets. But f is not almost-periodic since it is not uniformly bounded. We have $|f(x)| \leq C|\log(|x|)$, $|x| \geq 1$. Finally f is not mean-periodic.

5. Kahane's problem

In his 1957 paper Jean-Pierre Kahane raised a fundamental issue. He proposed the following problem: when do we have $\mathcal{B}_\Lambda = \mathcal{C}_\Lambda$? This leads to the following definition:

Definition 5.1. *A closed and discrete set $\Lambda \subset \mathbb{R}^n$ satisfies the property $Q(\Lambda)$ if $\mathcal{C}_\Lambda = \mathcal{B}_\Lambda$.*

A discrete set Λ for which property $Q(\Lambda)$ holds is called a *coherent set of frequencies* in [11].

Lemma 5.1. *Kahane's property $Q(\Lambda)$ implies that Λ is uniformly discrete.*

The converse implication is wrong as it will be seen below. An equivalent definition was given by Kahane in [5]:

Lemma 5.2. *$Q(\Lambda)$ is equivalent to the following condition: there exists a compact set K and a constant C such that for every finite trigonometric sum*

$$f(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x)$$

whose frequencies belong to Λ *one has*

$$\|f\|_\infty \leq C \sup_{x\in K} |f(x)|. \tag{4}$$

We write $Q(K,\Lambda)$ for the latter property. This implies that a function in $\mathcal{C}_\Lambda$ can be uniquely retrieved from its restriction to K with control on L^∞ norms. The next issue is, when possible, to replace K by a smaller compact set $K' \subset K$ for which $Q(K',\Lambda)$ still holds. This problem will be given a surprising solution in Theorem 15.4.

The following observations will not be needed in the proof which will be given in Section 15. They explain why Kahane's problem is related to irregular sampling.

Remark 5.1. *Kahane's problem can be rephrased as a problem of stable interpolation (Section 12). Stable interpolation arises in the mathematical theory of irregular sampling.*

In his pioneering work Kahane extended property $Q(\Lambda)$ to other functional norms. Let us define *Stepanov almost-periodic functions*. Let $1 \leq p \leq \infty$ and let L^p_{loc} be the vector space consisting of all measurable functions f such that $\|f\|_{B,p} = (\int_B |f|^p\, dx)^{1/p}$ is finite for every compact ball B. The topology of L^p_{loc} is defined by these semi-norms $\|\cdot\|_{B,p}$. Let $p\in[1,\infty]$ and $f\in L^p_{loc}$. The Stepanov norm of f is defined by

$$\|f\|_{S^p} = \sup_{x\in\mathbb{R}^n} \Big(\int_{|y-x|\leq 1} |f(y)|^p\, dy\Big)^{1/p}$$

when the right hand side is finite.

Definition 5.2. *With these notations f belongs to the Stepanov space S^p if and only if f is the limit with respect to the Stepanov norm $\|\cdot\|_{S^p}$ of a sequence of trigonometric polynomials.*

For instance the Heaviside function does not belong to S^p. The space S^∞ coincides with the Banach space $\mathcal{B}$ of almost-periodic functions in the sense of H. Bohr. Let $\Lambda\subset\mathbb{R}^n$ be uniformly discrete. On the one hand let us consider the closure $\mathcal{C}^p_\Lambda$ of trigonometric polynomials $f(x)=\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$ with respect to the topology of L^p_{loc}. On the other hand let us consider the closure S^p_Λ of trigonometric polynomials $f(x)=\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$ with respect to the Stepanov norm. Kahane asked in [5] whether or not one has $\mathcal{C}^p_\Lambda = S^p_\Lambda$. If it is the case we say that $Q(\Lambda,p)$ holds. An equivalent definition is given by the following proposition:

Proposition 5.1. *Let us assume $1\leq p\leq\infty$. Then property $Q(\Lambda,p)$ holds if and only if there exist a compact set K and a constant C such that for every trigonometric polynomial $P(x)=\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i x\cdot\lambda)$ with frequencies in Λ we have*

$$\sup_{y\in\mathbb{R}^n} \Big(\int_{K+y} |P(x)|^p\, dx\Big)^{1/p} \leq C\Big(\int_K |P(x)|^p\, dx\Big)^{1/p}. \tag{5}$$

If $p=2$ everything looks simpler and (5) is equivalent to

$$\Big(\sum_{\lambda\in\Lambda} |c(\lambda)|^2\Big)^{1/2} \leq C\Big(\int_K |P(x)|^2\, dx\Big)^{1/2}. \tag{5bis}$$

If Λ is uniformly discrete (5bis) holds if K is a sufficiently large ball [5]. If (5bis) holds for a compact set K it will hold for any L containing K. Given Λ we would like to know how small K can be. Landau's theorem (Section 13) implies that the Lebesgue measure of K cannot be smaller than the lower density of Λ. It will be proved in Section 15 that this necessary condition is sufficient (up to an arbitrarily small ϵ).

We return to $p=\infty$. Property $Q(\Lambda)$ depends on the arithmetical structure of the set Λ as Theorems 5.1 and 6.1 are showing.

Theorem 5.1. *Let $\alpha > 0$ be a real number. Then*

$$\Lambda_\alpha = -\mathbb{N} \cup \alpha\mathbb{N} = \{\ldots, -4,\ -3,\ -2,\ -1,\ 0,\ \alpha,\ 2\alpha,\ 3\alpha,\ 4\alpha, \cdots\}$$

satisfies Kahane's $Q(\Lambda)$ property if and only if $\alpha \in \mathbb{Q}$.

This theorem is not needed in this contribution and will be proved in a forthcoming paper. Another example is studied in the next section.

6. Pisot–Thue–Vijayaraghavan numbers

Definition 6.1. *A Pisot–Thue–Vijayaraghavan number is an algebraic integer $\theta > 1$ of degree n whose $n-1$ conjugates $\theta_2, \ldots, \theta_n$ satisfy $|\theta_j| < 1$.*

Natural integers $\theta \in \mathbb{N}$, $\theta \geq 2$, are Pisot–Thue–Vijayaraghavan numbers. Beyond natural integers the best known example is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. We have $\phi^2 - \phi - 1 = 0$. Its conjugate is $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ and $|\overline{\phi}| < 1$.

Definition 6.2. *Let $\theta > 1$ be a real number. We define Λ_θ as the set of all finite sums* $\sum_{k\geq 0} \epsilon_k \theta^k, \quad \epsilon_k \in \{0, 1\}$.

Let us return to Kahane's problem.

Theorem 6.1. *Let $\theta > 1$ be a real number. Then Λ_θ satisfies Kahane's property $Q(\Lambda)$ if and only if θ is a Pisot—Thue– -Vijayaraghavan number.*

Therefore Axel Thue's work is closely related to Kahane's problem.

Here is the proof of Theorem 6.1. One way is almost trivial. Let us assume that θ is not a Pisot number and prove that $Q(\Lambda_\theta)$ does not hold. We consider the sequence $P_m(x)$ of finite products $P_m(x) = \prod_0^{m-1}(\frac{1+\exp(2\pi i\theta^k x)}{2})$. By Pisot's theorem we know that $|P_m(x)| = \prod_0^{m-1} |\cos(\pi\theta^k x)|$ converge uniformly to 0 on every compact set not containing the origin. We have $P_m(0) = 1$. Assuming by contradiction that $Q(K, \Lambda_\theta)$ is true, we choose $x_0 \notin K$ and consider $R_m(x) = P_m(x - x_0)$. Then R_m converges uniformly to 0 on K while $R_m(x_0) = 1$. We have reached the required contradiction.

Let us sketch the proof of the converse implication. If θ is a Pisot number, then Λ_θ is contained in a model set [11] and every model set Λ satisfies Kahane's property $Q(\Lambda)$. We will return to this point in Section 15.

We have more.

Theorem 6.2. *Let $\theta > 2$ be a Pisot–Thue– Vijayaraghavan number. Then for every compact set K with a non empty interior there exists a constant C such that for every finite trigonometric sum $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda \cdot x)$ one has*

$$\|f\|_\infty \leq C \sup_{x\in K} |f(x)|. \tag{6}$$

This is proved in [10]. Here are two more examples. Proofs will appear elsewhere.

Theorem 6.3. *Let α and $\beta \neq 0$ be two real numbers. Then the set of real numbers*

$$\Lambda = \{k + \beta\sin(2\pi\alpha k),\ k \in \mathbb{Z}\}$$

satisfies Kahane's property $Q(\Lambda)$ if and only if $\alpha \in \mathbb{Q}$.

Let $\alpha > 1$ be irrational. Then the union $\Lambda = \mathbb{Z} \cup \alpha\mathbb{Z}$ does not satisfy $Q(\Lambda)$ since it is not a uniformly discrete set (Lemma 5.1). For a positive β let $\Lambda_{(\alpha,\beta)} \subset \Lambda$ be defined by deleting from Λ all integers $k \in \Lambda$ whose distance to $\alpha\mathbb{Z}$ is less than β and similarly deleting from $\alpha\mathbb{Z}$ all $\alpha k \in \alpha\mathbb{Z}$ whose distance to $\mathbb{Z}$ is less than β.

Theorem 6.4. *$\Lambda_{(\alpha,\beta)}$ satisfies Kahane's property $Q(\Lambda)$ but is not a model set.*

The definition of a model set is given in Section 15.

7. Local structure of almost-periodic functions

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. Let $\mathcal{B}_\Lambda$ be the corresponding space of almost-periodic functions. Then $\mathcal{B}_\Lambda$ is the closure with respect to topology of uniform convergence on $\mathbb{R}^n$ of the vector space of all finite trigonometric sums whose frequencies belong to Λ.

The *local structure* of $\mathcal{B}_\Lambda$ is studied by restricting $f \in \mathcal{B}_\Lambda$ to a compact set K. If K is "small" one cannot expect that $f \in \mathcal{B}_\Lambda$ can be retrieved from its restriction to K. We denote by $R(K, \Lambda)$ the extreme degeneracy: *Every continuous function on K is the restriction to K of a function $f \in \mathcal{B}_\Lambda$*. If it is the case the restriction to K of a function $f \in \mathcal{B}_\Lambda$ does not provide any information on f.

Definition 7.1. *We say that property $R(K, \Lambda)$ holds if every continuous function g on K is the restriction to K of an almost-periodic function belonging to $\mathcal{B}_\Lambda$.*

It then seems that Kahane's property $Q(K, \Lambda)$ is very distant from the degeneracy property $R(K, \Lambda)$. The following result provides us with an example where $Q(K, \Lambda)$ and $R(K, \Lambda)$ are almost contiguous. Let $|K|$ denote the Lebesgue measure of a compact set K.

Theorem 7.1. *Let $\Lambda \subset \mathbb{R}^n$ be a simple quasi-crystal and let $\beta = \operatorname{dens} \Lambda$. Then*

(a) *$R(K, \Lambda)$ holds for every compact set K such that $|K| < \beta$.*

(b) *Kahane's property $Q(K, \Lambda)$ holds if K is Riemann integrable and $|K| > \beta$.*

Simple quasi-crystals are defined in Section 15. Theorem 7.1 will follow from Lemma 15.3 and Theorem 15.4.

Remark 7.1. *Property $R(K, \Lambda)$ can be rephrased as a statement about stable sampling.*

This will be proved below (Lemma 11.2). The same problem can be raised for other functional norms. The following definition is seminal in the theory of irregular sampling.

Definition 7.2. *Let $K \subset \mathbb{R}^n$ be a compact set with positive Lebesgue measure. Then a uniformly discrete set Λ is a set of stable sampling for PW_K^2 if every function $f \in L^2(K)$ is the restriction to K of a Stepanov almost-periodic function $f \in S_\Lambda^2$.*

The Paley-Wiener space PW_K^2 will be defined in the next section and stable sampling will be studied in Section 11.

8. Paley-Wiener space

Definition 8.1. *Let $K \subset \mathbb{R}^n$ be a compact set. We write $f \in PW_K^2$ if $f \in L^2(\mathbb{R}^n)$ and if its Fourier transform $\widehat{f}$ vanishes almost everywhere outside K.*

If $n = 1$ and $K = [-\omega, \omega]$ then $f \in PW_K^2$ is a band limited signal and its cutoff frequency is ω.

Definition 8.1 can be generalized to L^p spaces.

Definition 8.2. *If $p \in [1, \infty]$ we write $f \in PW_K^p$ if $f \in L^p(\mathbb{R}^n)$ and if the distributional Fourier transform $\widehat{f}$ of f vanishes outside K.*

In satellite imaging K depends on the optics of the instrument. All images $f \in PW_K^2$ are sampled on a given lattice Γ. A lattice Γ is a discrete subgroup such that the quotient $\mathbb{R}^n/\Gamma$ is compact. Equivalently $\Gamma = A(\mathbb{Z}^n)$ where $A \in GL_n(\mathbb{R})$. Sampling an image $f \in PW_K^2$ on Γ yields the sequence $f(\gamma),\ \gamma \in \Gamma,$

belonging to $l^2(\Gamma)$. One needs to retrieve f from its samples. This is solved by the generalized Shannon-Nyquist theorem (Theorem 9.1). The choice of this lattice Γ is seminal in the economical success of the satellite. Coarse lattices are prohibited by the generalized Nyquist-Shannon theorem. Fine lattices are too expensive. The optimal lattice depends on the geometry of K. The definition of the constant c_Γ which is given below will be used in Section 9 and 15.

Definition 8.3. *If* $\Gamma = A(\mathbb{Z}^n)$ *is a lattice we set* $c_\Gamma = |det\, A|^{-1}$.

9. Generalized Shannon's theorem

The generalized Shannon-Nyquist's theorem governs the design of an optimal lattice in satellite imaging. Given a compact set $K \subset \mathbb{R}^2$ our goal is to sample images $f \in PW_K^2$ on a lattice Γ in the most efficient way. The dual lattice of a lattice $\Gamma \subset \mathbb{R}^n$ is defined by $\Gamma^* = \{y;\, \exp(2\pi i y \cdot x) = 1,\ \forall x \in \Gamma\}$. The generalized Shannon-Nyquist's theorem is true in any dimension.

Theorem 9.1. *The three following properties are equivalent*

(a) *The mapping* $S : PW_K^2 \mapsto l^2(\Gamma)$ *defined by* $S(f) = (f(\gamma))_{\gamma\in\Gamma}$ *is injective*

(b) *For every* $f \in PW_K^2$ *one has* $\|f\|_2 = c_\Gamma^{-1}(\sum_{\gamma\in\Gamma} |f(\gamma)|^2)^{1/2}$

(c) *For every* $\gamma^* \in \Gamma^*$ *one has*

$$\gamma^* \neq 0 \Rightarrow |(K + \gamma^*) \cap K| = 0. \tag{7}$$

Here $|E|$ denotes the Lebesgue measure of the Borel set E.

Y. Katznelson in [6] (Chapter IV, Interpolation of linear operators, page 113) constructed a compact set K such that the implication $(b) \Rightarrow (c)$ in Theorem 9.1 fails when PW_K^2 is replaced by PW_K^p with $1 \leq p < 2$. More precisely Katznelson proved the following:

Theorem 9.2. *There exists a compact set* $K \subset \mathbb{R}$ *of positive Lebesgue measure such that* $PW_K^p = \{0\}$ *for* $1 \leq p < 2$.

Theorem 9.2 is true in any dimension. Let $K \subset \mathbb{R}^n$ be the compact set constructed by Katznelson. Then the stable recovery property becomes trivial. But this does not imply necessarily (7). This pathology does not occur if K is Riemann integrable.

In satellite imaging one is given K and the issue is to find the sparsest Γ for which (7) holds. This corresponds to finding Γ^* as dense as possible. This has been achieved by Bernard Rougé who elaborated the SPOT 5 satellite. Satellite SPOT 5 was launched on April 2002 by an Ariane rocket. It was built by the CNES Agency. During fifteen years SPOT 5 provided Earth images with a resolution of 2.5 meters. SPOT 5 relied on a new sampling concept named "Supermode". The sampling grid Γ used in SPOT 5 was the sparsest one to be consistent with the optics K of the satellite.

10. Sparsity

A "regular grid" is a lattice. For a long time sampling a band limited signal on a regular grid was considered as a good fortune while an "irregular grid" was viewed as a wrong choice. To our greatest surprise the opposite is true. Sampling a band limited signal on a simple quasi-crystal circumvents the strong limitations imposed by Shannon-Nyquist's theorem.

This line of research is an illustration of the new paradigm of compressed sensing of sparse signals developed by Emmanuel Candès and David Donoho [2]. Compressed sensing is the following program. A collection $\mathcal{C}$ of signals or images have a sparse representation in a given orthonormal basis $\mathbf{B}$ if expansion of every $f \in \mathcal{C}$ in $\mathbf{B}$ only activates "a few vectors" depending on f. The compressed sensing program amounts to finding a universal *sparse collection* $\mathcal{G}$ of vectors such that every $f \in \mathcal{C}$ can be retrieved from the few samples $< f, g >$, $g \in \mathcal{G}$. This sparse collection cannot be contained in $\mathbf{B}$.

Definition 10.1. *Let f be a 1-D signal or a 2-D image and let K_f be the closure of the support of the Fourier transform $\widehat{f}$ of f. We say that f is sparse in the Fourier domain if the Lebesgue measure $|K_f|$ of K_f is small. Let β be a positive real number. One writes $f \in SF(\beta)$ if $|K_f| \leq \beta$.*

Definition 10.1 is consistent with Landau's theorem (Theorem 13.1). Landau's theorem relates the "sampling rate" to the Lebesgue measure of the spectrum. Let us observe that $f \in SF(\beta)$ and $g \in SF(\beta)$ imply $f + g \in SF(2\beta)$ since the measure of the union between K_f and K_g does not exceed 2β.

The paradigm of compressed sensing implies the following.

Claim 10.1. *For every positive β there exists a coarse grid Λ_β such that (a) the density of Λ_β tends to 0 with β and (b) every sparse signal $f \in SF(\beta)$ can be retrieved from its samples on Λ_β.*

Lattices do not work. Here is an example. Let us assume that $n = 2$, let $R > \epsilon > 0$ and let $K(\epsilon)$ be the annulus defined by $R - \epsilon \leq |x| \leq R$. Then $|K(\epsilon)| \simeq 2\pi R\epsilon$ as $\epsilon \to 0$. If the functions $f \in PW^2_{K(\epsilon)}$ are sampled on a lattice Λ the coarsest choice is the hexagonal lattice whose density is $2\sqrt{3}R^2$. The density of this lattice does not tend to 0 as $\epsilon \to 0$. Sampling on a quasi-crystal solves this problem. Indeed Theorem 15.1 will tell us that the optimal sampling rate only depends on the Lebesgue measure of the spectrum. More precisely for every $\beta > |K(\epsilon)|$ there exists a quasi-crystal Λ with density β which is a set of stable sampling for $PW^2_{K(\epsilon)}$. This relation between the "sampling rate" β of Λ and the Lebesgue measure $|K(\epsilon)|$ of the spectrum is optimal as Landau's theorem (Theorem 13.1) will show.

11. Stable sampling on arbitrary sets

Band-limited signals will now be sampled on uniformly discrete sets.

Definition 11.1. *Let $K \subset \mathbb{R}^n$ be a compact set and let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. Then the sampling operator $S : PW^2_K \mapsto l^2(\Lambda)$ is defined by $S(f) = (f(\lambda))_{\lambda\in\Lambda}$.*

The sampling operator S is continuous.

Definition 11.2. *Λ is a set of stable sampling for PW^2_K if the operator $S : PW^2_K \mapsto l^2(\Lambda)$ has a bounded left inverse: there exists a continuous operator $T : l^2(\Lambda) \mapsto PW^2_K$ such that $TS = I$.*

In other words Λ is a set of *stable sampling* for PW^2_K if there exists a constant C such that for every $f \in PW^2_K$ the following holds

$$\|f\|_2 \leq C(\sum_{\lambda\in\Lambda} |f(\lambda)|^2)^{1/2}. \tag{8}$$

Lemma 11.1. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set and $K \subset \mathbb{R}^n$ be a compact set. Then the following two properties are equivalent*

- (a) *Λ is a set of stable sampling for PW^2_K*
- (b) *Every function $f \in L^2(K)$ is the restriction to K of a function F belonging to the Stepanov space S^2_Λ.*

Using property (b) f can be written as $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$ and one has $(\sum_{\lambda\in\Lambda} |c(\lambda)|^2)^{1/2} \leq C(\int_K |f(x)|^2\, dx)^{1/2}$ for a constant C. This property was already introduced in Definition 7.2.

Stable sampling will now be addressed when 2 is replaced by $p \in [1, \infty]$. The space PW^p_K is defined by Definition 8.2. Let $p \in [1, \infty]$ and let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. Then the sampling operator $S : PW^p_K \mapsto l^p(\Lambda)$ is continuous.

Definition 11.3. *Λ is a set of stable sampling for PW^p_K if there exists a constant C such that for every $f \in PW^p_K$ the following holds*

$$\|f\|_p \leq C(\sum_{\lambda\in\Lambda} |f(\lambda)|^p)^{1/p}.$$

We then say that property $S(p, K, \Lambda)$ holds.

Property $R(K, \Lambda)$ was defined in Section 7. It can be rephrased as a statement about stable sampling where the space L^p is replaced by be the Banach algebra $B = B(\mathbb{R}^n)$ of Fourier-Stieltjes transforms $\widehat{\mu}$ of bounded complex Radon measures μ. Then $\widehat{\mu}$ is uniformly continuous and can be restricted to Λ. Let $B(\Lambda)$ be the corresponding restriction algebra equipped with the quotient norm (see [13]). Elements of $B(\Lambda)$ are sequences $c(\lambda)$, $\lambda \in \Lambda$, of the form $c(\lambda) = \widehat{\mu}(\lambda)$, $\lambda \in \Lambda$, where μ is a bounded complex Radon measure.

Definition 11.4. *Let $f \in B$ and let $\widehat{f}$ be its distributional Fourier transform. We write $f \in PW_K^B$ if the Radon measure $\widehat{f}$ is supported by K.*

Here the sampling operator $S : PW_K^B \mapsto B(\Lambda)$ is a contraction by the definition of the norm in $B(\Lambda)$.

Definition 11.5. *With these notations Λ is a set of stable sampling for PW_K^B if there exists a constant C such that for every $f \in PW_K^B$ we have*

$$\|f\|_B \leq C\|f\|_{B(\Lambda)}.$$

Lemma 11.2. *Then $R(K, \Lambda)$ is equivalent to the following condition: Λ is a set of stable sampling for PW_K^B.*

The proof can be found in [11], Chapter "Special Series", Lemma 10.

12. Stable interpolation

Definition 12.1. *Let $1 \leq p \leq \infty$. A uniformly discrete set Λ is a set of stable interpolation for PW_K^p if for every $c(\lambda)_{\lambda\in\Lambda} \in l^p$ there exists a function $f \in PW_K^p$ such that $f(\lambda) = c(\lambda)$, $\lambda \in \Lambda$. We then say that $T(p, K, \Lambda)$ holds.*

In Section 15 the case $p = \infty$ will play a seminal role. That is why the notation $T(\infty, K, \Lambda)$ will be simplified into $T(K, \Lambda)$.

If $K = [-\omega, \omega]$, the grid $h\mathbb{Z}$ is a set of stable sampling for PW_K^2 if and only if $0 < h \leq 1/(2\omega)$. It is the Shannon-Nyquist theorem. This grid is a set of stable interpolation for PW_K^2 if and only if $h \geq 1/(2\omega)$. This extends to $1 < p < \infty$. If $p = 1$ or $p = \infty$ stable interpolation requires $h > 1/(2\omega)$.

As announced in Remark 5.1, Kahane's property $Q(K, \Lambda)$ will be rephrased as a problem of stable interpolation where the space L^p is replaced by the Banach algebra $B(\mathbb{R}^n)$ which was defined in Section 11. Let us define stable interpolation in this new setting.

Definition 12.2. *Let $K \subset \mathbb{R}^n$ be a compact set and B the Banach algebra of Fourier-Stieltjes transforms of bounded complex Radon measures. A uniformly discrete set Λ is a set of stable interpolation for PW_K^B if every sequence belonging to the restriction algebra $B(\Lambda)$ is the restriction to Λ of a function belonging to PW_K^B.*

Lemma 12.1. *Kahane's property $Q(K, \Lambda)$ is equivalent to the following condition: Λ is a set of stable interpolation for PW_K^B.*

This is proved (with a different terminology) in [5], Proposition 3, page 298.

13. Landau's theorem

The following necessary conditions for sampling and interpolation were discovered in 1967 by H. J. Landau [7]. These conditions relate the uniform lower (resp. upper) density of a sampling set $\Lambda \subset \mathbb{R}^n$ to the Lebesgue measure of the spectrum.

Definition 13.1. *The uniform lower density of a closed and discrete set* $\Lambda \subset \mathbb{R}^n$ *is denoted by* $\underline{\mathrm{dens}}\ \Lambda$ *and is defined as follows. The Lebesgue measure of the ball* $B(x,R)$ *centered at* x *with radius* R *is* $|B(x,R)| = c_n R^n$*. Then one first computes* $N(R) = \inf_{x\in\mathbb{R}^n} \#[B(x,R)\cap\Lambda]$*. Finally*

$$\underline{dens}\ \Lambda = \liminf_{R\to\infty} \frac{N(R)}{c_n R^n}.$$

The uniform upper density is denoted by $\overline{\mathrm{dens}}\ \Lambda$ and is defined similarly: $N(R)$ is replaced by $M(R) = \sup_{x\in\mathbb{R}^n} \#[B(x,R)\cap\Lambda]$. Then

$$\overline{dens}\ \Lambda = \limsup_{R\to\infty} \frac{M(R)}{c_n R^n}.$$

Finally Λ has a uniform density if $\overline{\mathrm{dens}}\ \Lambda = \underline{\mathrm{dens}}\ \Lambda$. This common value is the density of Λ.

Let $|E|$ denote the Lebesgue measure of a Borel set $E \subset \mathbb{R}^n$ and let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set.

Theorem 13.1. *If* Λ *is a set of stable sampling for* PW_K^2 *we necessarily have*

$$\underline{\mathrm{dens}}\ \Lambda \geq |K|.$$

A similar theorem applies to stable interpolation.

Theorem 13.2. *If* Λ *is a set of stable interpolation for* PW_K^2 *we necessarily have*

$$\overline{\mathrm{dens}}\ \Lambda \leq |K|.$$

Landau's theorem does not hold if PW_K^2 is replaced by PW_K^p and $p \in [1,2)$ as Katznelson's theorem shows. However Landau's theorem is true for every $p \in [1,\infty]$ if K is Riemann integrable.

The converse implication:

$$\underline{\mathrm{dens}}\ \Lambda \geq |K| \Rightarrow \text{stable sampling} \tag{9}$$

does not hold in general. For instance (8) is wrong if Λ is a lattice. If Λ is a lattice the Shannon-Nyquist theorem (Theorem 9.1) imposes a constraint which is much stronger than $\underline{\mathrm{dens}}\ \Lambda \geq |K|$. Surprisingly (8) holds if (i) Λ is a simple quasi-crystal (Definition 15.2) and if (ii) $\underline{\mathrm{dens}}\ \Lambda \geq |K|$ is replaced by $\mathrm{dens}\ \Lambda > |K|$. This will be proved in Section 15, Theorem 15.1. We had to wait for fifty years to fully understand Landau's theorem. This was first achieved by Olevskii and Ulanovskii in [12] and then by Matei and the author in [8].

14. Universal sampling sets

Definition 14.1. *A universal sampling set for* L^2 *is a uniformly discrete set* $\Lambda \subset \mathbb{R}^n$ *enjoying the following properties*

- (a) Λ *has a uniform density which is strictly positive. This density is denoted by* $\mathrm{dens}\,\Lambda$.
- (b) Λ *is a set of stable sampling for* PW_K^2 *whenever the Lebesgue measure of the compact set* K *is strictly less than* $\mathrm{dens}\,\Lambda$.

This definition can be extended to other function spaces. The Paley-Wiener space PW_K^p is defined in Definition 8.2.

Definition 14.2. *Let* $p \in [1,\infty]$*. We say that* Λ *is a universal sampling set for* L^p *if*

- (a) Λ *has a uniform density which is strictly positive.*
- (b) Λ *is a set of stable sampling for* PW_K^p *whenever the Lebesgue measure of the compact set* K *is strictly less than* $\mathrm{dens}\,\Lambda$.

A. Olevskii and A. Ulanovskii proved the following theorem in [12] (Theorem 3.32, page 32):

Theorem 14.1. *The property: "*Λ *is a universal sampling set for* L^p*" does not depend on* $p \in [1,\infty]$.

From now on one writes "Λ is a universal sampling set" instead of "Λ is a universal sampling set for L^p."

In "A universal sampling of band-limited signals", C.R. Math. Acad. Sci. Paris, 342 (2006) 927-931, A. Olevskii and A. Ulanovskii proved the existence of universal sampling sets. It suffices to prove this existence theorem for $\operatorname{dens}\Lambda = 1$. The general case follows by a simple rescaling.

A subset of a lattice cannot be a universal sampling set. This implies that universal sampling sets have an intriguing arithmetical structure as indicated by the following observation.

Remark 14.1. *Let Λ be a universal sampling set. For every $\epsilon > 0$ there exists a uniformly discrete set Λ' which is obtained by moving each point of Λ by less than ϵ and which is not a universal sampling set.*

It suffices to replace each $\lambda \in \Lambda$ by $\lambda' \in \epsilon\mathbb{Z}^n$ such that $|\lambda' - \lambda| \le \frac{\sqrt{n}}{2}\epsilon$. Here is a stability result:

Lemma 14.1. *If $\Lambda = \{\lambda_j,\ j \in J\}$ is a universal sampling set then every perturbed set $\Lambda' = \{\lambda_j + r_j,\ j \in J\}$ such that $|r_j| \to 0$ as $j \to \infty$ is still a universal sampling set.*

It will be proved in a forthcoming paper. It implies that the elements of a universal sampling set can be linearly independent over $\mathbb{Q}$. Let $\alpha > 0$ be an irrational real number. Then the set $\Lambda_{(\alpha,\beta)}$ of Theorem 6.4 is not a universal sampling set. The proof of this fact will appear in a forthcoming paper. Richard Bass and Karlheinz Gröchenig [1] proved the following:

Theorem 14.2. *Let r_j, $j \in \mathbb{Z}$, be i.i.d. random variables equidistributed on $[0,1]$. Then almost surely the random set*

$$\Lambda = \{j + r_j;\ j \in \mathbb{Z}\}$$

is not a universal sampling set.

The density of Λ is 1. Bass and Gröchenig proved a more precise result:

Lemma 14.2. *Let $\alpha > 0$ and $K = [-\alpha, \alpha] \cup [1-\alpha, 1+\alpha]$. Then for almost all Λ there exists a sequence $f_k \in PW_K$ such that $\|f_k\|_2 = 1$ and $\sum_{\lambda\in\Lambda} |f_k(\lambda)|^2 \to 0,\ k \to \infty$.*

This random sequence r_j will now be replaced by a sequence $s_j \in [0,1]$ which is also equidistributed on $[0,1]$ but in a more even way than a random sequence. Then the resulting $\Lambda = \{j + s_j;\ j \in \mathbb{Z}\}$ can be a universal sampling set as the following example shows. Let $\{x\}$ be the *fractional part* of the real number x defined by $\{x\} \in [0,1)$ and $x - \{x\} \in \mathbb{Z}$.

Proposition 14.1. *Let us consider the increasing sequence of real numbers defined by$\lambda_j = j + \{j\sqrt{2}\};\ j \in \mathbb{Z}$, and $\Lambda = \{\lambda_j,\ j \in \mathbb{Z}\}$. Then Λ is a universal sampling set.*

This Λ resembles Bass-Gröchenig's random set. But $s_j = \{j\sqrt{2}\}$ is more evenly equidistributed than the random sequence r_j used in Bass-Gröchenig's theorem.

Let $\theta(x)$ be the distance from x to the nearest integer $k \in \mathbb{Z}$. Consider

$$M = \{j + \theta(j\sqrt{2});\ j \in \mathbb{Z}\}.$$

Is M a universal sampling set? M is the union between two disjoint model sets.

Our second example of a universal sampling set depends on a parameter $\alpha > 0$.

Proposition 14.2.

$$\Lambda_\alpha = \{m + n\sqrt{2};\ |m - n\sqrt{2}| \le \alpha,\ m, n \in \mathbb{Z}\}$$

is a universal sampling set.

The density of Λ_α is $\alpha/\sqrt{2}$.

A two dimensional example of a universal sampling set is described in the following proposition:

Proposition 14.3.
$$\Lambda_* = \{(k + \alpha\{k\sqrt{2} + l\sqrt{3}\}, l + \beta\{k\sqrt{2} + l\sqrt{3}\});\ k, l \in \mathbb{Z}\}$$
is a universal sampling set when α and β are two positive real numbers such that (a) α/β is irrational and (b) the three numbers $\alpha,\ \beta, 1 + \alpha\sqrt{2} + \beta\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

The density of Λ_* is 1.

These examples Λ, Λ_α, and Λ_* are three instances of simple quasi-crystals. Their properties will result of Theorem 15.1 which is proved in the next section.

15. Sampling on quasi-crystals

Simple quasi-crystals are defined now. Let $m, n \in \mathbb{N},\ N = m + n$, and $\Gamma \subset \mathbb{R}^N$ be a lattice: $\Gamma = A(\mathbb{Z}^N)$ where $A \in Gl_N(\mathbb{R})$. For $(x, t) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, we write $p_1(x, t) = x,\ p_2(x, t) = t$. Let us assume that p_1 once restricted to Γ is a 1-1 mapping with a dense range. The same is required on p_2.

Definition 15.1. *Let $I \subset \mathbb{R}^m$ be a compact set (a window). Then the model set $\Lambda_I \subset \mathbb{R}^n$ is defined by*
$$\Lambda_I = \{p_1(\gamma); \gamma \in \Gamma,\ p_2(\gamma) \in I\}.$$

A beautiful example of a 1-dimensional model set is given by the set $S(\mathcal{K}) \subset \mathbb{R}$ of all Pisot and Salem numbers of degree n belonging to an algebraic number field $\mathcal{K}$ of degree n over $\mathbb{Q}$. More precisely let $\mathcal{O}$ be the ring of all algebraic integers of $\mathcal{K}$ and let us denote by $p_1, \ldots, p_n$ the n embeddings of $\mathcal{K}$ into $\mathbb{R}$ or $\mathbb{C}$. Let us assume that p_1 is real. We then define a one dimensional model set $\Lambda \subset \mathbb{R}$ by
$$\Lambda = \{p_1(\omega);\ \omega \in \mathcal{O},\ |p_j(\omega)| \leq 1,\ 2 \leq j \leq n\}$$
This is exactly the definition of $S(\mathcal{K})$.

With this example in mind Pierre Deligne said that my model sets are obvious concepts for any mathematician who is fond of number theory. I agree. This example paves the way to the proof of Theorem 6.1. If θ is a Pisot number let us denote by $\theta_j,\ 2 \leq j \leq n$, the $n - 1$ conjugates of θ. If $\lambda \in \Lambda_\theta$ we have $\lambda = \sum_{k \geq 0} \epsilon_k \theta^k,\ \ \epsilon_k \in \{0, 1\}$. Then λ is an algebraic integer of the field $\mathcal{K}$ of degree n generated by θ. Moreover the conjugates $\lambda_j,\ 2 \leq j \leq n$, of λ (excluding λ itself) satisfy $|\lambda_j| \leq \frac{1}{1-\theta_j}$. Therefore Λ_θ is contained in a model set.

We now return to more general model sets.

Lemma 15.1. *Let $I \subset \mathbb{R}^m$ be a compact set. The upper density of the model set Λ_I is less than or equal to $c_\Gamma|I|$. If I is Riemann integrable the density of Λ_I equals $c_\Gamma|I|$.*

The constant c_Γ is defined by Definition 8.3. Lemma 15.1 is proved in [9], page 31.

Definition 15.2. *A model set Λ_I is a simple quasi-crystal if $m = 1$ and if I is a compact interval.*

The following theorem was proved in [8] and our goal is to provide the reader with a much simpler proof.

Theorem 15.1. *Let $\Lambda \subset \mathbb{R}^n$ be a simple quasi-crystal. Then Λ is a universal sampling set.*

In other words Λ is a stable sampling set for PW^2_K whenever the compact set $K \subset \mathbb{R}^n$ satisfies $|K| < \operatorname{dens}\Lambda$. Then, by Theorem 14.1, Λ is a stable sampling set for $PW^p_K,\ 1 \leq p \leq \infty$, whenever the compact set $K \subset \mathbb{R}^n$ satisfies $|K| < \operatorname{dens}\Lambda$. If $K \subset \mathbb{R}^n$ is a compact set such that $|K| > \operatorname{dens}\Lambda$, then Landau's theorem implies that Λ is not a stable sampling set for PW^2_K. The case $|K| = \operatorname{dens}\Lambda$ is studied in [4].

Three definitions will be used in our new proof of Theorem 15.1. As in Section 11 we write $f \in PW^p_K,\ 1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R}^n)$ and if its Fourier transform $\widehat{f}$ is supported by K. Then Λ is a stable sampling set for PW^p_K if there is a constant C such that $f \in PW^p_K$ implies

$$\|f\|_p \leq C(\sum_{\lambda\in\Lambda} |f(\lambda)|^p)^{1/p}. \tag{10}$$

Property (10) will be labeled $S(p, K, \Lambda)$. The case $p = \infty$ will be used in the proof of Theorem 15.1.

The fundamental property $R(K, \Lambda)$ was introduced in Definition 7.1. Property $R(K, \Lambda)$ is the following statement: *Every continuous function g on K is the restriction to K of an almost-periodic function f whose spectrum $\sigma(f)$ is contained in Λ.*

A third definition will be used in this context.

Definition 15.3. *Let $I \subset \mathbb{R}^m$ be a compact set and $M \subset \mathbb{R}^m$ be a uniformly discrete set. Property $T(I, M)$ is the following statement: Every sequence in $l^\infty(M)$ is the restriction to M of a function $F \in L^\infty(\mathbb{R}^m)$ whose Fourier transform $\widehat{F}$ is supported by I.*

Then $T(I, M)$ coincides with the property $T(\infty, I, M)$ of stable interpolation (Section 12). Arne Beurling discovered a simple necessary and sufficient condition ensuring $T(I, M)$ when $m = 1$ and when I is an interval (Theorem 15.3).

With these notations Theorem 15.1 follows from Lemma 15.2 and Lemma 15.3.

Lemma 15.2. *Let Λ be a uniformly discrete set and let us assume that three compact sets K, K', L satisfy the following conditions:*

(a) *K is contained in the interior of K'*

(b) *K' is contained in the interior of L.*

We then have

$$R(L, \Lambda) \Rightarrow S(\infty, K', \Lambda) \Rightarrow S(2, K, \Lambda) \tag{11}$$

Lemma 15.3. *Let Λ be a simple quasi-crystal. Then for every compact set L we have*

$$|L| < \operatorname{dens}\Lambda \Rightarrow R(L, \Lambda) \tag{12}$$

We now prove Theorem 15.1. Given K with $|K| < \operatorname{dens}\Lambda$ we can assume that K' and L satisfy (a) and (b) with $|L| < \operatorname{dens}\Lambda$. Since Theorem 15.1 can be rephrased as

$$|K| < \operatorname{dens}\Lambda \Rightarrow S(2, K, \Lambda) \tag{13}$$

it follows immediately from (11) and (12) as it was announced.

Let us begin with the proof of Lemma 15.2. We restate (10) as follows:

Proposition 15.1. *Let Λ be a uniformly discrete set and $1 \leq p \leq \infty$. If a compact set K is contained in the interior of a compact set L then $R(L, \Lambda)$ implies $S(p, K, \Lambda)$.*

Here is the proof of Proposition 15.1. As above we introduce a compact set K' which lies in between K and L. On the one hand K' is contained in the interior of L and we have $R(L, \Lambda) \Rightarrow S(\infty, K', \Lambda)$. This is Proposition 3, page 178, of [10]. On the other hand K is contained in the interior of K' and we have $S(\infty, K', \Lambda) \Rightarrow S(p, K, \Lambda)$. This is Theorem 3.32 by Olevskii and Ulanovskii in [12], page 32 (already stated as Theorem 14.1). Proposition 15.1 is proved.

We are left with proving Lemma 15.3 which will be rewritten as

$$|K| < \operatorname{dens}\Lambda \Rightarrow R(K, \Lambda) \tag{14}$$

where L is changed into K for notational convenience.

Lemma 15.3 will follow from the *duality principle* combined with Beurling's theorem (Theorem 15.3 below). Let us begin with the *duality principle* (Proposition 4, page 181 of [10]).

This *duality principle* will be stated as Theorem 15.2. The notations are given in Definition 15.1: $I \subset \mathbb{R}^m$ is an arbitrary compact set and $\Lambda_I \subset \mathbb{R}^n$ is the model set defined by this compact window I and the lattice Γ. As above $\Gamma^* = \{y \in \mathbb{R}^N; y \cdot x \in \mathbb{Z}, \forall x \in \Gamma\}$. The *duality principle* (Proposition 4, page 181 of [10]) is the following statement:

Theorem 15.2. *Let $K' \subset \mathbb{R}^n$ be a compact set and let $M_{K'} \subset \mathbb{R}^m$ be defined by*

$$M_{K'} = \{p_2(\gamma^*); \gamma^* \in \Gamma^*,\, p_1(\gamma^*) \in K'\}.$$

Let K be a compact set contained in the interior of K'. Then for every compact set $I \subset \mathbb{R}^m$ we have:

$$T(I, M_{K'}) \Rightarrow R(K, \Lambda_I). \tag{15}$$

The duality principle provides us with a duality between sampling and interpolation since $T(I, M_{K'})$ is a property of stable interpolation with respect to the $L^\infty(\mathbb{R}^m)$ norm while $R(K, \Lambda_I)$ is a property of stable sampling with respect to the $B(\mathbb{R}^m)$ norm (Lemma 11.2). A spectacular improvement on the duality principle with an outstanding application can be found in [4].

From now on $m = 1$ and I is a compact interval, which implies that $M_{K'}$ is a uniformly discrete set of real numbers.

Definition 15.4. *Let I be a compact interval. The Bernstein space $B_I = PW_I^\infty$ consists of all functions $F \in L^\infty(\mathbb{R})$ whose distributional Fourier transform is supported by I.*

Let us state the first half of Beurling's theorem [12] for the reader's convenience.

Theorem 15.3. *Let I be a compact interval and M a uniformly discrete set of real numbers. Then the following two properties are equivalent:*

- (a) *Every bounded sequence c_m, $m \in M$, is the restriction to M of a bounded function F belonging to the Bernstein space B_I.*
- (b) *The length of I is larger than the upper density of M.*

Here is the proof of Lemma 15.3. Beurling's theorem is seminal in this proof. As already stated the simple quasi-crystal Λ_I is defined by a window I which is an interval. By assumption we have

$$|K| < \operatorname{dens} \Lambda_I. \tag{16}$$

Lemma 15.1 gives

$$\operatorname{dens} \Lambda_I = c_\Gamma |I|. \tag{17}$$

Let K' be a compact set whose interior contains K and which still satisfies

$$|K'| < \operatorname{dens} \Lambda_I. \tag{18}$$

Let $M_{K'}$ be the "dual model set" defined by the lattice Γ^* and the window K'. Once more Lemma 15.1 yields

$$\overline{\operatorname{dens}}\, M_{K'} \le c_{\Gamma^*} |K'|. \tag{19}$$

But $c_{\Gamma^*}\, c_\Gamma = 1$. Then (17), (18) and (19) imply

$$\overline{\operatorname{dens}}\, M_{K'} < |I|. \tag{20}$$

Next Beurling's theorem yields property $T(I, M_{K'})$. Finally the duality principle gives

$$T(I, M_{K'}) \Rightarrow R(K, \Lambda_I) \tag{21}$$

which ends the proof of Lemma 15.3. Theorem 15.1 is now fully proved.

I was hoping that Theorem 15.1 could be extended to arbitrary model sets defined by arbitrary windows. Olevskii warned me that there was little hope to extend this proof to model sets defined by 2-dimensional windows since the proof of Theorem 15.1 is based on Beurling's theorem which is a 1-dimensional statement.

Olevskii was right. Here is a simple 2-dimensional counter example. Let $\Lambda_I \subset \mathbb{R}^2$ be the set of all $x = (k + \{k\sqrt{2}\}, m + \{m\sqrt{2}\}),\ (k,m) \in \mathbb{Z}^2$. Then Λ_I is a model set defined by the four-dimensional lattice

$$\Gamma = \{k + k\sqrt{2} - l,\ m + m\sqrt{2} - n,\ k\sqrt{2} - l,\ m\sqrt{2} - n;\ (k,l,m,n) \in \mathbb{Z}^4\} \tag{22}$$

and the window $I = [0,1]^2$.

This model set Λ_I is not a universal sampling set. Indeed Λ_I is the product $A \times A$ where A is a one dimensional model set. The density of A is 1. Let J be an interval of length larger than 1. Then Beurling's theorem implies the following: There exists a non trivial function g such that (a) g is supported by J and (b) its Fourier transform $\widehat{g}$ vanishes on A. Let h be a non trivial function supported by $[0,\epsilon]$. Then $f(x) = g(x_1)h(x_2)$ is supported by a compact set K_ϵ whose measure is arbitrarily small as ϵ tends to 0 while $\widehat{f}$ vanishes on Λ_I. Then Λ_I cannot be a stable sampling set.

Keeping the lattice defined by (22) we now consider an arbitrary compact window I. Then the corresponding model set Λ_I is contained in the product $A \times B$ between two one dimensional model sets A and B and cannot be a universal sampling set. The degeneracy does not come from the window but from the lattice. It can be conjectured that "generic model sets" are universal sampling sets. We do not know if the set of vertices of the Penrose paving is a universal sampling set.

We now return to the problem raised by Kahane. Property $Q(K,\Lambda)$ is defined by (4).

Theorem 15.4. *Let Λ be a simple quasi-crystal. Then for every Riemann integrable compact set K we have*

$$|K| > \operatorname{dens}\Lambda \Rightarrow Q(K,\Lambda). \tag{23}$$

Conversely $Q(K,\Lambda)$ implies $|K| \geq \operatorname{dens}\Lambda$ for every compact set K.

It would be interesting to know whether $|K| = \operatorname{dens}\Lambda \Rightarrow Q(K,\Lambda)$. Let us begin with the proof of the second statement. We argue by contradiction. If $|K| < \operatorname{dens}\Lambda$ then we denote by K' a compact set whose interior contains K and which still satisfies $|K'| < \operatorname{dens}\Lambda$. Lemma 15.3 yields $R(K',\Lambda)$. Let $a \in K' \setminus K$. Therefore there exists an almost periodic function g such that $g(a) = 1$, $g = 0$ on K and $g \in \mathcal{B}_\Lambda$. This contradicts $Q(K,\Lambda)$.

Theorem 15.4 cannot hold if Λ is a lattice. A simple counter example is given by $\Lambda = \mathbb{Z}$ and $K_N = [0,\epsilon] + \{0, 1, \cdots, N-1\}$. The measure of K_N tends to infinity with N and yet one cannot retrieve a $1-$periodic function from its restriction to K_N. Theorem 15.4 is not true for more general model sets. The counter example is the same as the one used for Theorem 15.1. The model set Λ_I is a product $A \times A$ where A is a one dimensional model set of density 1. Let J_1 be an interval of length smaller than 1 and let J_2 be an interval with a large length. Let $K = J_1 \times J_2$ with $|J_1| \cdot |J_2| > 1$. Let g_1 be a non trivial almost periodic function in one real variable, belonging to $\mathcal{B}_A$ and vanishing on J_1. Such g_1 exists by the argument used above. Finally the function F is defined by $F(x_1, x_2) = g_1(x_1)$. Then F belongs to $\mathcal{B}_{\Lambda_I}$ (since $0 \in A$) and vanishes on K without being the 0 function.

The proof of Theorem 15.4 is similar to the one used in Theorem 15.1. It combines the second half of the duality principle with the second half of Beurling's theorem. Let us be more precise.

Lemma 15.4. *Let $K \subset \mathbb{R}^n$ be a compact set and let M_K be the "dual model set" defined by*

$$M_K = \{p_2(\gamma^*); \gamma^* \in \Gamma^*,\ p_1(\gamma^*) \in K\}.$$

Then for every compact interval I we have

$$S(\infty, I, M_K) \Rightarrow Q(K, \Lambda_I).$$

This is an almost trivial statement in the duality principle [10]. Here is the second half of Beurling's theorem.

Theorem 15.5. *Let M be a uniformly discrete set of real numbers and let I be an interval. Then the following properties are equivalent:*

(a) *There exists a constant C such that for every function F belonging to the Bernstein space B_I one has*

$$\|F\|_\infty \leq C \sup_{m\in M} |F(m)|$$

(b) *The length of I is smaller than the lower density of M.*

As in the proof of Theorem 15.1 we have $|I| < \operatorname{dens} M_K$. Then Beurling's theorem yields $S(\infty, I, M_K)$. The duality principle ends the proof.

We can return now to the problem of stable interpolation.

Theorem 15.6. *Let Λ be a simple quasi-crystal and $\beta = \operatorname{dens}\Lambda$. Then Λ is a set of stable interpolation for PW_K^2 for every Riemann integrable compact set K such that $|K| > \beta$.*

Let us denote by K' and L two Riemann integrable compact sets such that (a) K' is contained in the interior of K, (b) L is contained in the interior of K' and (c) L still satisfies $|L| > \beta$. Then Kahane's property $Q(L, \Lambda)$ holds (Theorem 15.4). Proposition 3, page 178 of [10] yields $T(\infty, K', \Lambda)$. Finally we return to [12] and statement (i) of Theorem 4.11 ends the proof. The proof of Theorem 4.11 is written by Olevskii and Ulanovskii in the one dimensional case but it extends with minor changes to $\mathbb{R}^n$. Here also Theorem 15.6 is not true for more general model sets.

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